INSTRUCTIONS. Each problem is worth 20 points. Do as many as you can. Calculators may NOT be used.

1. Show that every group of order 992(= 32 × 31) is solvable.

2. Find the Galois group of the splitting field of the polynomial \( x^3 - x - 1 \) if the group field is
   (a) \( \mathbb{R} \)
   (b) \( \mathbb{Q} \)
   (c) \( \mathbb{Z}/2\mathbb{Z} \)
   (d) \( \mathbb{Z}/5\mathbb{Z} \)
   (e) \( \mathbb{Z}/23\mathbb{Z} \)

3. List all abelian groups of order 200. Each group on your list should be displayed as a direct product of cyclic groups, and no two groups on the list should be isomorphic.

4. Are the \( 4 \times 4 \) matrices
   \[
   A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}
   \]
   similar? Explain your reasoning.

5. Determine all ideals of the ring \( \mathbb{Z}[x]/(2, x^3 + 1) \).

6. Fermat proved that the number \( 2^{37} - 1 = 137438953471 \) was composite by finding a small prime factor \( p \). Suppose you know \( 200 < p < 300 \). What is \( p \)?

7. Determine the minimal polynomial of the element \( \alpha = \sqrt{2} + \sqrt{5} \). (In other words, find \( \text{Irr}(\alpha, \mathbb{Q}) \).)

8. (a) Prove carefully that the rings \( \mathbb{Q}[x]/(x^2 - 2) \) and \( \mathbb{Q}[x]/(x^2 - 3) \) are not isomorphic.
   (b) Find an example of a commutative ring with unity \( R \) such that \( R[x]/(x^2 - 2) \) and \( R[x]/(x^2 - 3) \) are isomorphic. Justify your answer briefly.

9. Let \( \zeta \) be a primitive 13th root of unity. For each element \( \alpha \) listed below, find \( [\mathbb{Q}(\alpha) : \mathbb{Q}] \).
   (a) \( a = \zeta \)
   (b) \( \alpha = \zeta + \zeta^{12} \)
   (c) \( \alpha = \zeta + \zeta^2 \)

10. An \( R \)-module \( M \) is said to be irreducible if \( M \neq 0 \) and \( M \) has no submodules except 0 and \( M \). Let \( V \) be a finite-dimensional vector space over a field \( k \), and let \( T : V \to V \) be a linear transformation. We know that \( T \) gives \( V \) the structure of a \( k[x] \)-module. Prove that \( V \) is irreducible as a \( k[x] \)-module if and only if the characteristic polynomial of \( T \) is irreducible in \( k[x] \).