Phd Exam Fall 2002

Work at least 7 problems from the real analysis section and at least 3 from the complex analysis section.

Real Analysis

1. Give an example of a measure space, $(\Omega, \mu, \mathcal{F})$, and a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function $f$, such that inequality is obtained in Fatou’s lemma.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to [-\infty, \infty]$ are measurable. Prove the sets
   \[ \{ \omega : f(\omega) < g(\omega) \} \quad \text{and} \quad \{ \omega : f(\omega) = g(\omega) \} \]
   are measurable. Note you can’t add or subtract functions which have values in this space and expect the operations to be continuous.

3. Let $E$ be a countable subset of $\mathbb{R}$. Show $m(E) = 0$.

4. Given $1 > \varepsilon > 0$, show there exists an open set $E \subseteq [0, 1]$ dense in $[0, 1]$, and $m(E) = \varepsilon$. \textbf{Hint:} Recall the construction of the Cantor set. Next show there exists a strictly increasing function, $f$, which has the property that its derivative equals zero on a set of positive measure.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) \equiv (1 + |x|^2)^k$. Find the values of $k$ for which $f$ is in $L^1(\mathbb{R}^n)$. \textbf{Hint:} Use polar coordinates.

6. Let $B$ be a Borel set in $\mathbb{R}^n$ and let $v$ be a nonzero vector in $\mathbb{R}^n$. Suppose $B$ has the following property. For each $x \in \mathbb{R}^n$, $m(\{t : x + tv \in B\}) = 0$. Then show $m_n(B) = 0$. Note the condition on $B$ says roughly that $B$ is thin in one direction.

7. If $f : \mathbb{R}^n \to [0, \infty]$ is Lebesgue measurable, show there exists $g : \mathbb{R}^n \to [0, \infty]$ such that $g = f$ a.e. and $g$ is Borel measurable.

8. Suppose $E$ is a Lebesgue measurable set which has positive measure and let $B$ be an arbitrary open ball and let $D$ be a set dense in $\mathbb{R}^n$. Establish the result of Smítal, which says that under these conditions, $\overline{m_n}((E + D) \cap B) = m_n(B)$ where here $\overline{m_n}$ denotes the outer measure determined by $m_n$. Is this also true for $X$, an arbitrary possibly non measurable set replacing $E$ in which $\overline{m_n}(X) > 0$? \textbf{Hint:} Let $x$ be a point of density of $E$ and let $D'$ denote those elements of $D$, $d$, such that $d + x \in B$. Thus $D'$ is dense in $B$. Now use translation invariance of Lebesgue measure to verify there exists, $R > 0$ such that if $r < R$, we have the following holding for $d \in D'$ and $r_d < R$.
   \[ \overline{m_n}((E + D) \cap B (x + d, r_d)) \geq m_n((E + d) \cap B (x + d, r_d)) \geq (1 - \varepsilon) m_n(B (x + d, r_d)). \]
   Argue the balls, $m_n(B (x + d, r_d))$, form a Vitali cover of $B$.

9. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set
   \[ E - E = \{ x - y : x \in E, y \in E \}. \]
   Show that $E - E$ contains an interval. \textbf{Hint:} Let
   \[ f(x) = \int \chi_E(t) \chi_E(x + t) dt. \]
   Note $f$ is continuous at $0$ and $f(0) > 0$. Remember continuity of translation in $L^p$. 


10. Suppose for all \( f \in C_{c}(0, \infty) \), \( \|Af\|_{L_{p}(0,\infty)} \leq K\|f\|_{L_{p}(0,\infty)} \) where \( A \) is a linear operator defined on \( L_{0}^{p}(0,\infty) \). Does this inequality hold for all \( f \in L_{0}^{p}(0,\infty) \)? Explain why or why not. Here it is understood that the measure is ordinary Lebesgue measure.

11. Let \( f \in L_{1}^{1}(\mathbb{R}^{n}) \). Show \( Mf \), the Maximal function, is Borel measurable. Recall

\[
Mf(x) \equiv \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x)| \, dx.
\]

**Complex Analysis**

1. It is desired to find an analytic function, \( L(z) \) defined for all \( z \in \mathbb{C} \setminus \{0\} \) such that \( e^{L(z)} = z \). Is this possible? Explain why or why not.

2. If \( f \) is analytic, show that \( z \to f(\bar{z}) \) is also analytic.

3. Let \( f : U \to \mathbb{C} \) be analytic and \( f(z) = u(x,y) + iv(x,y) \). Show \( u, v \) and \( uv \) are all harmonic although it can happen that \( u^2 \) is not. Recall that a function, \( w \) is harmonic if \( w_{xx} + w_{yy} = 0 \).

4. Suppose that for some constants \( a, b \neq 0, a, b \in \mathbb{R} \), \( f(z+ib) = f(z) \) for all \( z \in \mathbb{C} \) and \( f(z+a) = f(z) \) for all \( z \in \mathbb{C} \). If \( f \) is analytic, show that \( f \) must be constant. Can you generalize this? **Hint:** This uses Liouville’s theorem.

5. Suppose \( f \) is an entire function and that \( f \) has the property that whenever we write \( f(z) \) as a power series expanded about a point \( w \), it follows that at least one of the coefficients in the power series must equal zero. Show that \( f \) must be a polynomial. **Hint:** Define a set, \( A_{n} \) to be the points, \( w \) such that if \( f(z) = \sum_{k=0}^{\infty} a_{k} (z-w)^{k} \), it follows \( a_{n} = 0 \). Thus \( A_{n} \) consists of the points where the power series of \( f \) centered at these points has the \( n^{th} \) coefficient equal to zero. Argue that some \( A_{n} \) is uncountable and therefore has a limit point.

6. We say a real valued function, \( u \) is subharmonic if \( u_{xx} + u_{yy} \geq 0 \). Show that if \( u \) is subharmonic on a bounded region, (open connected set) \( U \), and continuous on \( \overline{U} \) and \( u \leq m \) on \( \partial U \), then \( u \leq 0 \) on \( U \). State and prove a theorem about the uniqueness of the solutions to the equation, \( u_{xx} + u_{yy} = 0 \) in \( U \) and \( u = f \) on \( \partial U \). **Hint for the first part:** If not, \( u \) achieves its maximum at \((x_{0},y_{0}) \in U \). Let \( u(x_{0},y_{0}) > m + \delta \) where \( \delta > 0 \). Now consider \( u_{\varepsilon}(x,y) = \varepsilon x^{2} + u(x,y) \) where \( \varepsilon \) is small enough that \( 0 < \varepsilon x^{2} < \delta \) for all \((x,y) \in U \). Show that \( u_{\varepsilon} \) also achieves its maximum at some point of \( U \) and therefore, \( u_{xx} + u_{yy} \leq 0 \) at that point implying that \( u_{xx} + u_{yy} \leq -\varepsilon \), a contradiction.

7. Use Rouche’s theorem to prove the fundamental theorem of algebra which says that if \( p(z) = z^{n} + a_{n-1}z^{n-1} + \cdots + a_{1}z + a_{0} \), then \( p \) has \( n \) zeros in \( \mathbb{C} \). **Hint:** Let \( q(z) = -z^{n} \) and let \( \gamma \) be a large circle, \( \gamma(t) = re^{it} \) for \( r \) sufficiently large.

8. Prove Liouville’s theorem from the Cauchy integral formula.