1. Let $M$ be a smooth manifold, $TM$ its tangent bundle, and $\pi: TM \to M$ the bundle map. Prove the extension lemma for vector fields:

Let $Y$ be a vector field defined on a closed subset $A \subset M$ (so $Y: A \to TM$ is a map satisfying $\pi \circ Y = Id_A$ and for all $p \in A$, there exists a neighborhood $V_p$ of $p$ in $M$ and a smooth vector field $\tilde{Y}$ on $V_p$ that agrees with $Y$ on $V_p \cap A$). If $U$ is an open set containing $A$, show there exists a smooth vector field $\tilde{Y}$ on all of $M$ such that $\tilde{Y}|_A = Y$ and the support of $\tilde{Y}$ is contained in $U$.

2. Prove that for any 2–manifold $M$ smoothly embedded in $\mathbb{R}^6$, there exists $v \in S^5$ such that orthogonal projection in the direction of $v$ gives an injective map of $M$ to a hyperplane.

3. Let $M$ be a smooth manifold. Suppose $\gamma_0, \gamma_1: [0, 1] \to M$ are smooth curves that are path homotopic. For every closed 1–form $\omega$ on $M$, prove:

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

4. Show the complement of a finite set of points in $\mathbb{R}^n$ is simply connected if $n \geq 3$.

5. Define a $\Delta$–complex structure on a Klein bottle $K$ and use it to compute the homology groups of $K$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.

6. Let $M$ be a closed, orientable $n$–manifold. Let $F^i$ denote $H^i(M; \mathbb{Z})$ with torsion factored out.

(a) Prove that $F^i$ is isomorphic to $\text{Hom}(H^{n-i}(M), \mathbb{Z})$. What is the isomorphism? That is, for $\alpha \in F^i$, the isomorphism takes $\alpha$ to a homomorphism sending $\phi \in H^{n-i}(M)$ to which element of $\mathbb{Z}$?

(b) If $\alpha$ generates a $\mathbb{Z}$–summand of $H^{n-i}(M; \mathbb{Z})$, prove there exists $\beta \in H^i(M; \mathbb{Z})$ so that the cup product $\alpha \cup \beta$ generates $H^n(M; \mathbb{Z})$. 