Please be aware: some of the questions are tricky, and some of the questions are tricks.

Chapter 1 Concepts

1) $A$ is an $m \times n$ matrix, and $B$ is a $p \times q$ matrix. For each of the following, what must be true about $m, n, p,$ and $q$ for the operation to be defined?

i) $A + B$  
ii) $AB$  
iii) $BA$  
iv) $A^T B$  
v) $B^T A$

2) A box contains 15 coins. There are pennies, nickels, and dimes in the box, but no other type of coin. The total value of the coins in the box is 77 cents. Determine how many coins of each type may be in the box. (Bonus: Find ALL possible combinations that satisfy the problem.)

3) Let

$$
\begin{pmatrix}
  a & b & 2 & 8 \\
  b & 1 & 1 & a \\
  0 & b & 2 & 4
\end{pmatrix}
$$

be the augmented matrix for a linear system. For what values of $a$ and $b$ does the system have

a) a unique solution  
b) a one-parameter solution  
c) a two-parameter solution  
d) no solution

4) Find a matrix $K$ such that $AKB = C$ if

$$
A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix},
B = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix},
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

5) Let $A$ be a square matrix. Show $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$. 
Chapter 2 Concepts

1) Let $A$ be a $3 \times 3$ matrix, each of whose entries is either 1 or 0. What is the largest possible value for $\det(A)$?

2) Prove: If $A$ is an invertible matrix, then $\text{adj}(A)^{-1} = \text{adj}(A^{-1})$.

3) Solve by Cramer’s rule:
   \[
   \begin{align*}
   \pi x_1 - 2x_2 + 3x_3 - 4x_4 &= -4 \\
   3x_1 - x_2 - 5x_3 + 5x_4 &= -2 \\
   4x_1 + x_2 - 2x_3 + x_4 &= 2 \\
   x_1 + x_2 - x_3 - \sqrt{3}x_4 &= 2
   \end{align*}
   \]

Chapter 3 Concepts

1) Let $u = (-1, 1, 1)$, $v = (1, -1, 1)$, and $w = (1, 1, -1)$.
   
   Compute the following:
   \[
   \begin{align*}
   &a) \ u + v + w \\
   &b) \ u \cdot (v \times w) \\
   &c) \ (u \cdot v) \times w \\
   &d) \ (u \times v) \times w \\
   &e) \ u \times (v \times w) \\
   &f) \ |2u + 3v - w|
   \end{align*}
   \]

2) Describe, geometrically, the set of all vectors in $\mathbb{R}^2$ that are orthogonal to a given nonzero vector. What about in $\mathbb{R}^3$?

3) True of False: If a vector is orthogonal to itself, then it is orthogonal to any other vector in the same space.

Chapter 4 Concepts

1) Define a vector space consisting of regular polygons (A line may be considered as a 2 sided polygon, a circle as 1 sided), identify the $0$ object.
   a) If we only allow scalars to be integers, then prove that this is a vector space (list each vector space axiom and explain why it holds).
   b) Prove that the set of regular polygons with an even number of sides is a subspace.
2) Describe, geometrically in $R^3$, the solution space of the given system for arbitrary values of $s$.

\[
\begin{align*}
    x_1 + & x_2 + sx_3 = 0 \\
    x_1 + & sx_2 + x_3 = 0 \\
    sx_1 + & x_2 + x_3 = 0
\end{align*}
\]

3) Find a basis for $P_n$ such that every polynomial in the basis has degree $n$.

4) Explain what the following symbols are:
   a) $R^2$  b) $M_{nn}$  c) $P_n$  d) $C(-\infty, \infty)$

5) Draw two linearly dependent vectors in $R^2$.

6) Use the Wronskian to determine whether $f_1 = \sqrt{2}, f_2 = e^x$, and $f_3 = 2e^x + \pi$ are linearly independent.

7) If $v = (3, 2, 1)$ is the coordinate vector of $u$ relative to the basis $S$, find $u$.
   a) $S = \{1, x, x^2\}$
   b) $S = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$
   c) $S = \{(1, 1), (1, -1), (0, 1)\}$

Problems 8-10 refer to each other.

8) Create 3 distinct bases for $R^3$ and prove that they are bases. Call these $S_1$, $S_2$, and $S_3$.

9) Create the transition matrix to the standard basis of $R^3$ from each of your bases in 8. Call these $P_1$, $P_2$, and $P_3$.

10) Express the transition matrix from $S_1$ to $S_2$ in terms of $P_1$, $P_2$, $P_3$ and their inverses.

11) True/False: If $B$ is the row echelon form of $A$, then
   a) $B$ and $A$ have the same row space.
   b) $B$ and $A$ have the same null space.
   c) $B$ and $A$ have the same column space.
   d) The columns that form a basis for the column space of $B$ correspond to those that do so for $A$.

12) Write the standard matrix $A$ for orthogonal projection on the $xy$-plane in $R^3$. What is the column space of $A$?
13) Find a basis for the row space, column space, and null space of the following matrix:
\[
\begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 0 & 1 & 1 \\
3 & 1 & 9 & 1 \\
0 & 1 & 0 & -5 \\
2 & 1 & 7 & 0
\end{bmatrix}
\]

14) Find a basis for the orthogonal complement of \{(1, 0, 0, -1), (0, 1, 0, -5), (0, 0, 1, 1)\}.

**Chapter 5 Concepts**

1) Show that if \(0 < \theta < \pi\) then the standard matrix for rotation by \(\theta\) in \(\mathbb{R}^2\) has no real eigenvectors. Explain what this means geometrically.

2) Prove: If \(A\) is diagonalizable, with nonnegative eigenvalues, then there is a matrix \(S\) such that \(S^2 = A\).

3) If \(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\) is diagonalizable, what is the value of \(b\)?

4) The 3×3 matrix \(A\) has eigenvalues \(\lambda = a, b, c\) and corresponding eigenvectors \(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\). Find an expression for \(A\) as the product of invertible and diagonal matrices.

5) The matrix \(A\) has the following characteristic polynomial: \(\lambda^5 - \lambda^4 + \pi \lambda^3 - e\lambda^2 - \sqrt{2}\lambda + \ln(5)\). What is the determinant of \(A\)?

6) If \(A\) has eigenvalue \(\lambda\) and corresponding eigenvector \(x\), what do we know about the eigenvalues and eigenvectors of \(A^{-1}\), assuming \(A\) is invertible?

7) Prove your claim from 6.

**Chapter 6 Concepts**

1) Define a weighted inner product on \(\mathbb{R}^2\) so that the "unit circle" is an ellipse \(a\) units wide along the \(x\)-axis and \(b\) units along the \(y\)-axis.
2) Determine whether $\langle \mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{v}, A^T\mathbf{u} \rangle$ if $\langle \mathbf{u}, \mathbf{v} \rangle$ is the Euclidean inner product on $\mathbb{R}^n$ and $A$ is $n \times n$.

3) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.

Find the least squares solution to the linear equation $A\mathbf{x} = \mathbf{b}$. Find the least squares error vector. Verify that it is orthogonal to the column space of $A$.

4) Transform the basis $(1, -3), (2, 2)$ into an orthonormal basis. Find the coordinate vector of $(1, 1)$ with respect to that basis.

5) Suppose $a, b,$ and $c$ are points on a circle, such that $a$ and $c$ lie on the diameter of the circle. Prove that the angle $abc$ is a right angle.

**Chapter 7 Concepts**

1) Let $A = \begin{bmatrix} a & 1 & -1 \\ b & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\ c & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{3} \end{bmatrix}$.

Find all values $a, b,$ and $c$ so that $A$ is orthogonal.

2) If $A$ is any $m \times n$ matrix, is it true that $A^T A$ is orthogonally diagonalizable? Prove your answer.

3) Let $A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$.

Find a matrix $P$ that orthogonally diagonalizes $A^{-1}$ WITHOUT calculating $A^{-1}$.

4) Find an upper triangular matrix that is orthogonally diagonalizable.

5) If an $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) and corresponding, orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, write an expression to calculate $A$. 
6) Let $A$ be a $3 \times 3$ matrix.
a) If $A$ has eigenvectors $(0,1,2), (3,0,0),$ and $(0,-6,3)$, is $A$ symmetric?
b) If $A$ has the same eigenvectors as above, and corresponding eigenvalues $-1, 3,$ and $0$, find $A$. 