1 1.1 - Basic Math Models

Easy
Model a differential equation where all solutions are asymptotic to \( y = 3 - t \) as \( t \to \infty \)

Medium 22
Create a differential model to find the volume in the following situation:

A spherical raindrop evaporates at a rate proportional to its surface area.

Hard 24
A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm\(^3\) of the drug enters the patient’s bloodstream at a rate of 100 cm\(^3\)/hr. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (hr\(^{-1}\)). Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation that models how much of the drug is present in the bloodstream at any given time.
2 2.1 - Integrating Factors

Easy 2
Solve the following ODE
\[ y' - 2y = t^2 e^{2t} \]

Medium
Solve the following ODE
\[ ty' - y = te^{-t}, \quad t > 0 \]

Hard 20
Solve the following IVP
\[ ty' + (t + 1)y = t, \quad y(\ln 2) = 1, \quad t > 0 \]
3 2.3 - First Order ODE Modelling

Easy 2
A tank has initially 120 L of pure water. A mixture containing a concentration of $\gamma$ g/L of salt enters the tank at a rate of 2 L/min. The well-stirred mixture leaves at the same rate. Determine how much salt is in the tank at any given time.

Medium
A tank contains 200 gal of water with 50 lbs of salt. Water with 1 lb of salt per gallon enters at a rate of 4 gal/min, and the mixture leaves the tank at 2 gal/min. How much salt is in the tank when there are 400 gallons of water?

Hard 13
The mosquito population increases at a rate proportional to the current population and doubles each week. There are 200,000 mosquitoes in the area initially, and predators eat 20,000 mosquitoes/day. Determine how many mosquitoes are in the area at any given time.
4 2.5 - Autonomous Equations

Easy 2
Find all of the equilibrium points and classify them as stable, unstable, or semi-stable.

(a) \( dy/dt = ay + by^2, \quad a > 0, \quad b > 0, \quad -\infty < y_0 < \infty \)

(b) \( dy/dt = k(1 - y)^2y(y + 3) \)

Medium 8, 10
Repeat the "Easy" instructions, draw the phase line, and sketch several graphs of solutions in the ty-plane.

(a) \( dy/dt = -k(y - 1)^2, \quad k > 0, \quad -\infty < y_0 < \infty \)

(b) \( dy/dt = y^2(1 - y)^2, \quad -\infty < y_0 < \infty \)

Hard 16
Suppose a population follows the model,
\[
dy/dt = ry \ln(K/y),
\]
where \( r \) and \( K \) are positive constants. Sketch \( f(y) \) versus \( y \), find the critical points, and determine whether each is asymptotically stable or unstable.
5 2.6 - Exact Equations

Easy 7
Determine if the following differential equation is exact. If it is, find the solution.

\[(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0\]

Medium 16
Find a value of \( b \) for which the given equation becomes exact, and solve for the solution.

\[(ye^{2xy} + x)dx + bxe^{2xy}dy = 0\]

Hard 29
Make the following Differential Equation exact.

\[e^x dx + (e^x \cot y + 2y \csc y)dy = 0\]
6  2.8 - Existence and Uniqueness Theorem

**Easy**
State the existence and uniqueness theorem for first-order differential equations. (Both linear and non-linear versions).

**Medium 2**
Transform the given initial value problem into an equivalent problem with the initial point at the origin.

\[
\frac{dy}{dt} = 1 - y^3, \quad y(-1) = 3.
\]

**Hard 6**
Using successive approximations, solve the initial value problem,

\[
y' = y + 1 - t, \quad y(0) = 0
\]
7 3.1 - Homogeneous Equations with Constant Coefficients

Easy 15
Solve the following IVP.

\[ y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0 \]

Medium 18
Find an ODE where all possible solutions can be represented by
\[ y = c_1 e^{-t/2} + c_2 e^{-2t} \]

Hard 23
Determine the values of \( \alpha \), if any, for which all solutions tend to zero as \( t \to \infty \); also determine values, if any, for which all non-zero solutions become unbounded as \( t \to \infty \).

\[ y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0 \]
8 3.2 - Wronskian

Easy 9
Determine the longest interval in which a solution exists for the following ODE.

\[ t(t - 4)y'' + 3ty' + 4y = 2 \]

Medium 28
Show that

\[ e^{-t}, \quad e^{2t} \]

form a fundamental set of solutions to \( y'' - y' - 2y = 0 \).

Hard 35
If \( y_1 \) and \( y_2 \) are a fundamental set of solutions to \( t^2y'' - 2y' + (3 + t)y = 0 \) and if \( W(y_1, y_2)(2) = 3 \), find the value of \( W(y_1, y_2)(4) \).
9 3.3 - Complex Roots

**Easy 22**
Solve the following IVP:

\[ y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2 \]

**Medium 29**
Show that \( \cos t = \frac{e^{it} + e^{-it}}{2} \) and \( \sin t = \frac{e^{it} - e^{-it}}{2i} \)

**Hard 40**
Use the Euler Technique to transform the following ODE into one with constant coefficients and find the general solution.

\[ t^2 y'' - ty' + 5y = 0 \]
10  3.4 - Repeated Roots

**Easy 13**
Solve the following IVP:

\[9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2\]

**Medium 27**
Solve for a second homogeneous solution for the following ODE, given that \(y_1(x) = \sin x^2\).

\[xy'' - y' + 4x^3y = 0, \quad x > 0\]

**Hard 46**
Solve the following ODE using Euler’s technique.

\[t^2y'' + 5ty' + 13y = 0, \quad t > 0\]
11 3.5 - Undetermined Coefficients

Easy 13
Solve the following IVP:

\[ y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1 \]

Medium
Prove that the solution \( \phi(t) \) to a non-homogeneous second order ODE can be written as

\[ \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \]

where \( y_1(t), y_2(t) \) are homogeneous solutions and \( Y_1(t) \) is a particular solution (Theorem 3.5.2).

Hard 25
Find an appropriate undetermined coefficients form for the particular solution of the following differential equation.

\[ y'' + 3y' + 2y = e^t (t^2 + 1) \sin(2t) + 3e^{-t} \cos t + 4e^t \]
12  3.6 - Variation of Parameters

Easy 7
Solve the following ODE:

\[ y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0 \]

Medium 18
Find the complete solution to the following ODE, given that the homogeneous solutions are \( y_1(x) = x^{-1/2}\sin x \), and \( y_2(x) = x^{-1/2}\cos x \).

\[ x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x, \quad x > 0 \]

Hard 31
Use reduction of order to find the complete solution to the following ODE given that one homogeneous solution is \( y_1(t) = 1 + t \).

\[ ty'' - (1 + t)y' + y = t^2 e^{2t}, \quad t > 0 \]
13 3.7 - Mechanical and Electrical Vibrations

Easy 4
Write $u = -2 \cos(\pi t) - 3 \sin(\pi t)$ in the form $u = R \cos(\omega_0 t - \delta)$

Medium 13
A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position $u$ at any time $t$. Find the quasi frequency $\mu$ and the ratio of $\mu$ to the natural frequency of the corresponding undamped motion.

Hard 20
Assume that the system described by the equation $mu'' + \gamma u' + ku = 0$ is critically damped and that the initial conditions are $u(0) = u_0, u'(0) = v_0$. If $v_0 = 0$, show that $u \to 0$ as $t \to \infty$ but that $u$ is never zero. If $u_0$ is positive, determine a condition on $v_0$ that will ensure that the mass passes through its equilibrium position after it is released.
14 3.8 - Forced Vibrations

Easy 4
Write $\sin 3t + \sin 4t$ as a product of two trigonometric functions of different frequencies.

Medium 9
If an undamped spring-mass system with a mass that weighs 6 lb and a spring constant 1 lb/in is suddenly set in motion at $t = 0$ by an external force of $4 \cos 7t$ lb, determine the position of the mass at any time and draw the graph of the displacement versus $t$.

Hard 17
Consider a vibrating system described by the initial value problem

$$u'' + \frac{1}{4} u' + 2u = 2 \cos \omega t, \quad u(0) = 0, \quad u'(0) = 2$$

Find the steady state part of the solution of the problem. At what frequency $\omega$ will this system experience resonance? What kinds of frequencies will result in a beat?
15 5.1 - Review of Power Sets

Easy 8
Find the radius of convergence of
\[
\sum_{n=1}^{\infty} \frac{n!x^n}{n^2}
\]

Medium 27
Rewrite the following expression as a sum whose generic term involves \( x^n \).
\[
x \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n
\]

Hard 28
Determine the \( a_n \) so that the equation
\[
\sum_{n=1}^{\infty} na_nx^{n-1} + 2 \sum_{n=0}^{\infty} a_nx^n = 0
\]
is satisfied. Try to identify the function represented by the series \( \sum_{n=0}^{\infty} a_nx^n \).
16 5.2 - Series Solutions Near an Ordinary Point, Part I

Easy 8
Find the first four terms in each of the two solutions, $y_1, y_2$.

$$xy'' + y' + xy = 0, \quad x_0 = 1$$

And then evaluate the Wronskian at the point $x_0$.

Medium 10
Find a series solution to the following ODE:

$$(4 - x^2)y'' + 2y = 0, \quad x_0 = 0$$

Hard 19
Find a series solution to the following ODE:

$$y'' + (x - 1)^2y' + (x^2 - 1)y = 0$$

by first making the change of variable $x - 1 = t$. 
17 5.3 - Series Solutions Near an Ordinary Point, Part II

**Easy 3**
Determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{(4)}(x_0)$ for the point $x_0 = 1$ if $y = \phi(x)$ is a solution of

$$x^2y'' + (1 + x)y' + 3(\ln x)y = 0; \quad y(1) = 2, \quad y'(1) = 0$$

**Medium 7**
Determine a lower bound for the radius of convergence of series solutions about each given point $x_0$ for

$$(1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2$$

**Hard 13**
Find a series solution to the following ODE:

$$\cos x)y'' + xy' - 2y = 0$$

*Hint:* Use the identity $m!a_m = \phi^{(m)}(x_0)$
5.4 - Euler Equations; Regular Singular Points

**Easy 8**
Determine the general solution of \( 2x^2y'' - 4xy' + 6y = 0 \)

**Medium 39**
Consider the Euler equation \( x^2y'' + \alpha xy' + \beta y = 0 \). Find conditions on \( \alpha \) and \( \beta \) so that:

(a) All solutions approach zero as \( x \to 0 \)
(b) All solutions approach zero as \( x \to \infty \)
(c) All solutions are bounded both as \( x \to 0 \) and as \( x \to \infty \).

**Hard 40**
Use the method of reduction of order to show that if \( r_1 \) is a repeated root of

\[
r(r - 1) + \alpha r + \beta = 0,
\]

then \( x^{r_1} \) and \( x^{r_1} \ln x \) are solutions of \( x^2y'' + \alpha xy' + \beta y = 0 \) for \( x > 0 \).
19 5.5 - Series Solutions Near a Regular Singular Point, Part I

Easy 5
Determine the indicial equation, recurrence relation, and roots of the indicial equation for the following ODE.

\[ 3x^2y'' + 2xy' + x^2y = 0 \]

Medium 10
Find the series solution to the following ODE using the greater root of the indicial equation:

\[ x^2y'' + \left(x^2 + \frac{1}{4}\right)y = 0 \]

Hard 8
Find the series solution to the following ODE using the greater root of the indicial equation:

\[ 2x^2y'' + 3xy' + (2x^2 - 1)y = 0 \]
20  5.6 - Series Solutions Near a Regular Singular Point, Part II

Easy 7
Find all the regular singular points, the indicial equation, and the exponents at the singularity for each regular singular point of the following ODE:

\[ x^2 y'' + \frac{1}{2} (x + \sin x)y' + y = 0 \]

Medium 17
Find the exponents at the singular point \( x = 0 \) and the first three nonzero terms in each of the two solutions (not multiples of each other) of the following differential equation:

\[ x^2 y'' + (\sin x)y' - (\cos x)y = 0 \]

Hard 18
Determine the roots of the indicial equation of

\[ (\ln x)y'' + \frac{1}{2} y' + y = 0 \]

at \( x = 1 \). Then determine the first three nonzero terms in the series \( \sum_{n=0}^{\infty} a_n (x - 1)^{r + n} \) corresponding to the larger root. Take \( x - 1 > 0 \).
6.1 - Definition of the Laplace Transform

Easy 7
Find the Laplace transform of $\cosh bt$.

Medium 13
Find the Laplace transform of $e^{at} \sin bt$.

Hard 19
Find the Laplace transform of $t^2 \sin at$
22  6.2 - Solution of Initial Value Problems

Easy 8
Find the inverse Laplace transform of

\[ F(s) = \frac{8s^2 - 4s + 12}{s(s^2 + 4)} \]

Medium 17
Use the Laplace Transform to solve the given initial value problem:

\[ y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 0, \quad y^{(3)}(0) = 1 \]

Hard 24
Find the Laplace transform \( Y(s) = L\{y\} \) of the solution of the given initial value problem.

\[ y'' + 4y = \begin{cases} 1, & 0 \leq t < \pi, \\ 0, & \pi \leq t < \infty; \end{cases} \quad y(0) = 1, \quad y'(0) = 1 \]
23 6.3 - Step Functions

Easy 12
Express

\[ f(t) = \begin{cases} 
  t, & 0 \leq t < 2 \\
  2, & 2 \leq t < 5 \\
  7 - t, & 5 \leq t < 7 \\
  0, & t \geq 7.
\end{cases} \]

in terms of the unit step function \( u_c(t) \).

Medium 23
Find the inverse Laplace transform of

\[ F(s) = \frac{(s - 2)e^{-s}}{s^2 - 4s + 3} \]

Hard 32
Find the Laplace transform of

\[ f(t) = 1 - u_1(t) + \ldots + u_{2n}(t) - u_{2n+1}(t) = 1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t) \]
24  6.4 - Differential Equations with Discontinuous Forcing Functions

Easy 6
Find the solution to the ODE.

\[ y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1 \]

Medium 13
Find the solution to the ODE.

\[ y^{(4)} + 5y'' + 4y = 1 - u_\pi(t); \quad y(0) = y'(0) = y''(0) = y'''(0) = 0 \]

Hard 18
Consider the initial value problem

\[ y'' + \frac{1}{3}y' + 4y = f_k(t), \quad y(0) = y'(0) = 0. \]

where

\[ f_k(t) = \begin{cases} 
1/2k, & 4 - k \leq t < 4 + k \\
0, & 0 \leq t < 4 - k \text{ and } t \geq 4 + k 
\end{cases} \]

Solve the initial value problem and describe how the solution depends on \( k \).
25 6.5 - Impulse Functions

Easy 5
Find the solution to the ODE.

\[ y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); \quad y(0) = y'(0) = 0. \]

Medium 9
Find the solution to the ODE.

\[ y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \quad y(0) = y'(0) = 0. \]

Hard 24
Suppose that

\[ y'' + y = f(t), \quad y(0) = y'(0) = 0 \]

Predict the nature of the solution of this IVP given that

\[ f(t) = \sum_{k=1}^{15} \delta[t - (2k - 1)\pi] \]

And then determine what happens after the sequence of impulses ends.
Easy 3
Use \( \sin t \) as an example to show that \( f \ast f \) is not necessarily nonnegative.

Medium 10
Find the inverse Laplace transform of
\[
\frac{1}{(s + 1)^2(s^2 + 4)}.
\]

Hard 18
Find the solution of the following ODE
\[
y'' + 3y' + 2y = \cos \alpha t
\]
and express your answer in terms of the convolution integral.
7.1 - Introduction to Systems of First Order Linear Equations

Easy 4
Transform \( u^{(4)} - u = 0 \) in a system of first order linear differential equations.

Medium 12
Solve the following system of first order linear differential equations by first transforming it into a single second order differential equation.

\[
\begin{align*}
    x_1' &= -0.5x_1 + 2x_2, \quad x_1(0) = -2 \\
    x_2' &= -2x_1 - 0.5x_2, \quad x_2(0) = 2
\end{align*}
\]

Hard 22
Consider the following two interconnected tanks. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

Create a system of equations that models the amount of salt in each tank.
28  7.4 - Basic Theory of Systems of First Order Linear Equations

Easy 5
Show that the general solution of $x' = P(t)x + g(t)$ is the sum of any particular $x^{(p)}$ of this equation and the general solution $x^{(c)}$ of the corresponding homogeneous equation.

Medium 3
Show that the Wronskians of two fundamental sets of solutions of the system $x' = P(t)x$ can differ at most by a multiplicative constant. *Hint:* Use Abel’s Theorem.

Hard 2
Let $x^{(1)}$ and $x^{(2)}$ be solutions of $x' = P(t)x$ for $\alpha < t < \beta$, and let $W$ be the Wronskian of $x^{(1)}$ and $x^{(2)}$. Show that

$$\frac{dW}{dt} = \begin{vmatrix} \frac{dx^{(1)}}{dt} & \frac{dx^{(2)}}{dt} \\ x^{(1)}_2 & x^{(2)}_2 \end{vmatrix} + \begin{vmatrix} x^{(1)}_1 & x^{(2)}_1 \\ \frac{dx^{(1)}_2}{dt} & \frac{dx^{(2)}_2}{dt} \end{vmatrix}.$$
29 7.5 - Homogeneous Linear Systems with Constant Coefficients

Easy 1
Find the general solution of the given system.
\[ x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x \]

Medium 17
Solve the given initial value problem.
\[ x' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \]

Hard 19
The system \( tx' = Ax \) is analogous to the second order Euler equation. Assuming that \( x = \xi t^r \), where \( \xi \) is a constant vector, show that \( \xi \) and \( r \) must satisfy \( (A - rI)\xi = 0 \) in order to obtain nontrivial solutions of the given differential equation.
30  7.6 - Complex Eigenvalues

Easy 6
Express the general solution of the given system of equations in terms of real-valued functions.
\[ x' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} x \]

Medium 8
Express the general solution of the given system of equations in terms of real-valued functions.
\[ x' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} x \]

Hard 9
Find the solution of
\[ x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
31 7.7 - Fundamental Matrices

Easy
Find a fundamental matrix for
\[ x' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} x \]
and then find the fundamental matrix \( \Phi(t) \) satisfying \( \Phi(0) = I \)

Medium 11
Solve the initial value problem
\[ x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \]
using the fundamental matrix \( \Phi(t) \) where \( \Phi(0) = I \).

Hard 15
Let \( \Phi(t) \) denote the fundamental matrix satisfying \( \Phi' = A\Phi, \Phi(0) = I \). An alternate notation is \( \exp(At) \). Show that \( \Phi(t)\Phi(s) = \Phi(t+s) \); that is, show that \( \exp(At)\exp(As) = \exp(A(t+s)) \). Hint: Show that if \( s \) is fixed and \( t \) is variable, then both \( \Phi(t)\Phi(s) \) and \( \Phi(t+s) \) satisfy the initial value problem \( Z' = AZ, Z(0) = \Phi(s) \).
32 7.8 - Repeated Eigenvalues

Easy 2
Draw a direction field and sketch of few trajectories of

\[ x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x \]

and then find the general solution of the system.

Medium 6
Find the general solution of

\[ x' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \]

Hard 15
Show that all solutions of the system

\[ x' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \]

approach zero as \( t \to \infty \) if and only if \( a + d < 0 \) and \( ad - bc > 0 \).
33 7.9 - Nonhomogeneous Linear Systems

**Easy 3**
Solve the given system.

\[
x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}
\]

**Medium 5**
Solve the given system.

\[
x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x + \begin{pmatrix} t^{-3} \\ t^{-2} \end{pmatrix}, \quad t > 0
\]

**Hard 12**
Solve the given system

\[
x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}, \quad \frac{\pi}{2} < t < \pi
\]
34 9.1 - The Phase Plane: Linear Systems

Easy 1
Find the eigenvalues and eigenvectors. Then classify the critical point (0,0) and determine whether it is stable, asymptotically stable, or unstable.

\[
\frac{dx}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x
\]

Medium 14
Determine the critical point \( x = x^0 \), and then classify its type and examine its stability by making the transformation \( x = x^0 + u \).

\[
\frac{dx}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -2 \\ 1 \end{pmatrix}
\]

Hard 16
Determine the critical point \( x = x^0 \), and then classify its type and examine its stability by making the transformation \( x = x^0 + u \).

\[
\frac{dx}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} x + \begin{pmatrix} \alpha \\ -\gamma \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta > 0
\]
9.2 - Nonlinear Differential Equations and Stability

Easy 6
Find all the critical points, determine whether each critical point is asymptotically stable, stable, or unstable, and classify it to type.

\[
\frac{dx}{dt} = 1 + 2y, \quad \frac{dy}{dt} = 1 - 3x^2
\]

Medium 16
Find all the critical points, determine whether each critical point is asymptotically stable, stable, or unstable, and classify it to type.

\[
\frac{dx}{dt} = x(2 - x - y), \quad \frac{dy}{dt} = (1 - y)(2 + x)
\]

Hard 22
Find an equation of the form \( H(x, y) = c \) satisfied by the trajectories

\[
\frac{dx}{dt} = 2x^2y - 3x^2 - 4y, \quad \frac{dy}{dt} = -2xy^2 + 6xy
\]
36  9.3 - Locally Linear Systems

Easy 2
Verify that (0,0) is a critical point, show that the system is locally linear, and discuss the type and stability of the critical point (0,0) by examining the corresponding linear system.

\[
\frac{dx}{dt} = -x + y + 2xy, \quad \frac{dy}{dt} = -4x - y + x^2 - y^2
\]

Medium 7
Determine all the critical points and find the corresponding linear system near each one.

\[
\frac{dx}{dt} = 1 - y, \quad \frac{dy}{dt} = x^2 - y^2
\]

Hard 18
Determine all the critical points and find the corresponding linear system near each one.

\[
\frac{dx}{dt} = (1 - y)(2x - y), \quad \frac{dy}{dt} = (2 + x)(x - 2y)
\]
37 9.4 - Competing Species

Easy 2
For each critical point of the system, find the corresponding linear system. Determine the limiting behavior of \( x \) and \( y \) as \( t \to \infty \).

\[
\frac{dx}{dt} = x(1.5 - x - 0.5y), \quad \frac{dy}{dt} = y(2 - 0.5y - 1.5x)
\]

Medium 3
For each critical point of the system, find the corresponding linear system. Determine the limiting behavior of \( x \) and \( y \) as \( t \to \infty \).

\[
\frac{dx}{dt} = x(1.5 - 0.5x - y), \quad \frac{dy}{dt} = y(2 - y - 1.125x)
\]

Hard 7
Consider the eigenvalues given by

\[
r_{1,2} = -\left(\sigma_1X + \sigma_2Y\right) \pm \sqrt{\left(\sigma_1X + \sigma_2Y\right)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY}.
\]

Show that

\[
(\sigma_1X + \sigma_2Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY = (\sigma_1X - \sigma_2Y)^2 + 4\alpha_1\alpha_2XY
\]
38 9.5 - Predator-Prey Equations

**Easy 1**
For each critical point of the system, find the corresponding linear system. Determine the limiting behavior of \(x\) and \(y\) as \(t \to \infty\).

\[
\frac{dx}{dt} = x(1.5 - 0.5y), \quad \frac{dy}{dt} = y(-0.5 + x)
\]

**Medium 5**
For each critical point of the system, find the corresponding linear system. Determine the limiting behavior of \(x\) and \(y\) as \(t \to \infty\).

\[
\frac{dx}{dt} = x(-1 + 2.5x - 0.3y - x^2), \quad \frac{dy}{dt} = y(-1.5 + x)
\]

**Hard 9**
Consider the system

\[
\frac{dx}{dt} = ax[1 - (y/2)], \quad \frac{dy}{dt} = by[-1 + (x/3)],
\]

where \(a\) and \(b\) are positive constants. Suppose the initial conditions are \(x(0) = 5\) and \(y(0) = 2\). Let \(a = 1\) and \(b = 1\). Plot the trajectory in the phase plane and determine (or estimate) the period of the oscillation.
39  9.6 - Liapunov’s Second Method

Easy 2
Construct a suitable Liapunov function of the form $ax^2 + cy^2$. Then show that the critical point at the origin is asymptotically stable.

$$\frac{dx}{dt} = -\frac{1}{2}x^3 + 2xy^2, \quad \frac{dy}{dt} = -y^3$$

Medium 4
Construct a suitable Liapunov function of the form $ax^2 + cy^2$. Then show that the critical point at the origin is unstable.

$$\frac{dx}{dt} = x^3 - y^3, \quad \frac{dy}{dt} = 2xy^2 + 4x^2y + 2y^3$$

Hard 5
Consider the system of equations

$$\frac{dx}{dt} = y - xf(x, y), \quad \frac{dy}{dt} = -x - yf(x, y)$$

where $f$ is continuous and has continuous first partial derivatives. Show that if $f(x, y) > 0$ in some neighborhood of the origin, then the origin is an asymptotically stable critical point. Hint: Construct a Liapunov function of the form $c(x^2 + y^2)$.