334 Review

1. Solve $y' = x \sin^2(x) \cos(x)$ subject to $y(0) = 1$.

2. If $y_h$ and $y_p$ are solutions to the equations $y' + p(t)y = 0$ and $y' + p(t)y = g(t)$, respectively, show that $y = y_h + y_p$ is also a solution to $y' + p(t)y = g(t)$.

3. Solve the equation $y' = 2ty^2$ subject to the initial condition $y(0) = y_0$. Fully simplify the solution and state where/when the solution is valid.

4. Draw the phase portrait, phase line, and the solution plot of $y' = ay - b\sqrt{y}$ for $0 < a < b$ and $y(0) \geq 0$ and classify all equilibrium solutions without solving the equation. Do the same for $y' = y^2(y - 3)^3(y + 2)^4(y + 5)^{10}(y - 6)$.

5. Solve $p(t)y' + f(t)y = g(t)$ in all possible cases of $p(t), f(t), g(t)$ (You may assume that everything is continuous, integrable, and differentiable).

6. Solve $u' = -k(u - T_0 - T_1 \cos(\omega t))$ where $k, T_0, T_1, \omega$ are fixed constants.

7. Solve
   
   (a) $\sin(x)y' + \cos(x)y = 0$
   (b) $\left(3x + \frac{6}{y}\right) + \left(x^2 + \frac{3y}{x}\right) \frac{dy}{dx} = 0$
   (c) $(x^3 + x^2 - x - 1 + y)dx + (x^2 + x)dy = 0$
   (d) $2x^2dy + (3xy + 1)dx = 0$

8. Solve $(x - \sin(y))dy + \tan(y)dx = 0$ subject to $(1, \pi/6)$.

9. Solve $y(x) + 2 \int_0^x ty(t)dt = x^2$.

10. Draw the direction field for $y' = e^{-t}y + 3$.

11. Write down a differential equation such that
    
    (a) All solutions approach $y = 3$
    (b) All solutions diverge from $y = 2$
    (c) All solutions approach $y = t - 1$
    (d) All solutions diverge from $y = 2t + 1$

12. Solve $y' = ky(M - y)$ for some constants $k, M$ where $M$ satisfies $0 < y < M$ and $y(0) = y_0$. Determine the behavior of the solution as time increases in an unbounded fashion. You may assume $k > 0$.

13. Find the interval in which the solution of the differential equation $(\ln(t))y' - \csc(t)y = \frac{1}{\sqrt{\sqrt{2} - t}}$, $y(2) = 3$ exists.

14. A tank with a capacity of 1000 gallons originally contains 100 gallons of water with 200 pounds of salt. Water containing 3 pounds of salt per gallon enters the tank at a rate of 10 gallons per minute. The salt and water instantly mixes and the brine solution flows out of the tank at a rate of 8 gallons per minute. Write an equation to describe the amount of salt in the tank at any time and use this to find the concentration of salt (in pounds per gallon) in the tank when it is on the point of overflowing.
15. You start a college savings account for your future son or daughter, so you put away $5000. Your bank compounds your investment continuously at 6% and you deposit $1200 every year on January 1 because of the bonus you receive every year. Because you are so financially savvy, you never withdraw any money from the account. After ten years, you change banks so that you get a better rate of return, namely 7%. Furthermore, you get a promotion in your job and deposit $2000 a year for the next eight years. When your child turns 18, how much money will you be able to gift to your child for college?

16. You love to experiment on rats and rats populate very quickly. After one day of letting the rats populate, there are 1000 rats. After three days, there are 3000 rats. Assuming the rats populate continuously, don’t die for at least the length of your experiment, you have enough food and large enough area to sustain the growth in population, what is the reproduction rate and how many rats did you start with?

17. Solve \( t^2y' + 3ty = e^t \) subject to \( y(1) = 1 \).

18. Solve \((e^y + xy e^{xy} + \sin(y)) + (x^2 e^{xy} + x \cos(y) - 2y)y' = 0\).

19. Find a particular solution of \( y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t) \).

20. If \( y_1 \) and \( y_2 \) are solutions of \( 3t^2 y'' + 6ty' + 4 \sin(t) = 0 \) for \( t > 0 \), what could be the Wronskian of \( y_1 \) and \( y_2 \)?

21. A weight of 64 lbs stretches a spring 4 ft.
   (a) If the spring-mass system is in a viscous medium, what damping constant \( \gamma \) is necessary for critical damping?
   (b) If the system is in a medium where the damping force is 96 lbs when the object is at a velocity of 2ft/s, and if the mass on the spring is lifted 6 inches to then be released, how does the object move?

22. If \( y(t) \) is a solution to \( t^2 y'' + 4ty' + 2y = 0 \) for \( t > 0 \), determine whether or not the limit \( \lim_{t \to \infty} y(t) \) converges or not. If it doesn’t exist or doesn’t converge, say so. If it converges, calculate the limit.

23. Find the solution to the differential equation

   \[ y'' + 4y' + 5y = 2\delta(t - \pi/4) \]

   subject to \( y(0) = 0 \) and \( y'(0) = 1 \).

24. For \( t > 0 \), find the general solution to

   \[ t^2 y'' + 2ty' - 2y = \frac{1}{t^2}. \]

25. Find the inverse Laplace transform to

   \[ F(s) = \sum_{n=0}^{\infty} (-1)^n \frac{e^{-ns}}{s} \]

   and sketch the solution.
26. The Fourier series for the $2\pi$ periodic function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < 0 \\ x & \text{for } 0 \leq x < \pi \end{cases}$$

is given by

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(nx) + \frac{(-1)^n + 1}{n} \sin(nx).$$

What is the value of

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(n\pi).$$

27. Find two linearly independent solutions to $y'' + x^2 y = 0$.

28. An object that weighs 1.1 pounds stretches a spring 10 cm. The object moves through a viscous medium imparting 8N on the object at 4m/s. The machine applying the external force is turned off and when we start modeling the object's motion, there is a downward velocity of 2m/s. Formulate the initial value problem, but don’t bother to solve it.

29. Find the solution to $y'' - y' - 2y = \sin(t)$ subject to $y(0) = 0$ and $y'(0) = 1$.

30. Find a solution to $x'(t) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} x(t)$ subject to $x(0) = (2, 0, 1)^T$.

31. Find a solution to $x'(t) = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} x(t)$ subject to $x(0) = (1, 1)^T$.

32. Given $x'(t) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x(t)$, find a fundamental matrix satisfying $\Phi(0) = I$.

33. Initially the first tank contains 100 gallons of pure water, the second tank contains eighty gallons of pure water, and the third tank contains sixty gallons of pure water. Salt solution containing two pounds per gallon magically appears in the tank at a rate of five gallons per minute. Salt solution drains from the first tank at a rate of five gallons per minute into the second tank, the second tank at a rate of five gallons per minute into the third tank, and the third tank at a rate of five gallons per minute into a pot. Write a system in matrix notation describing the system.

34. Find a fundamental set of solutions to $x'(t) = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} x(t)$. Then, find a fundamental matrix satisfying $\Phi(0) = I$.

35. Find a solution to $\sin(x)y' + \cos(x)y = \frac{e^{ix} - e^{-ix}}{2i}$ subject to $y(\frac{3\pi}{4}) = 2$.

36. Show that the equation $N_1 - BN = kN^2$, $N(0) = N_0$ is solved by

$$N = \frac{BN_0 e^{-(Bt+N_0k)}}{(B-N_0k)}.$$  

Then, show that $N(t) \to 0$ as $t \to \infty$ for all $N_0 \neq 0$.  

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37. Find the general solution to \( x^2y'' + 2xy' - 2y = x \) valid in the domain \((0, \infty)\).

38. Find the Laplace Transform of \((t + a)^n\).

39. What is the definition of linear independence and how do you check for it?

40. Show that the general solution of \( x'(t) = (Px + g)(t) \) is the sum of the particular and homogeneous solution.

41. Solve the set of equations \( x_1' = x_1 - 2x_2 \) and \( x_2' = 3x_1 - 4x_2 \) simultaneously subject to \( x_1(0) = -1 \) and \( x_2(0) = 2 \).

42. Write the equation \( y^{(4)} - y = 0 \) as a system of first order equations in matrix notation.
Food for Thought

Solving equations is a tantalizing, yet tricky topic. It depends on many different things. My first two questions are “What order is it?” and “How many equations are there?” closely followed by “What is my path of least resistance in finding a solution?” If the answer to my questions are one and one, then I have a first order equation and can use separation, an integrating factor, a Laplace transform, a series solution, undetermined coefficients, or exact equation methods (which ever will be easiest). Keep in mind that separation means get one variable of one side and the other variable on the other. The integrating factor relies on an equation of the form $y' + p(x)y = g(x)$ and the integrating factor is $\exp\left[\int p(x)dx\right]$. You then multiply it through, integrate, and solve if applicable (recall that the purpose of the integrating factor is to make the left hand side a product rule). If I want the exact-ness, check $M_y = N_x$ which is Clairaut’s Theorem from multi-variable calculus. If $M_y = N_x$, just integrate both, union them, set it to $C$ and go on with your life. If not, then use $\exp\left[\int (M_y - N_x)/N dx\right]$, $\exp\left[\int (N_x - M_y)/M dy\right]$, or $\exp\left[(N_x - M_y)/(xM - yN)dx\right]$ to get an integrating factor of $x, y,$ or $(xy)$ respectively; multiply it through and check exactness again. If it is exact, use process one. If not, then check your integration because it should work. The Laplace Transfrom is, well, the then best option in my opinion because it can be done fairly quickly if the stuff isn’t to bad.

If the answer is some other order and one equation, then I start to think about spitting my solution into homogeneous and inhomogeneous OR using a Laplace Transform given some initial conditions (Note that if you don’t have any, make them up by saying $y(0) = a_0, y'(0) = a_1, ..., y^{(n)}(0) = a_n$ for however many you need and it will work; I don’t claim it will be easy, but it will work). Now you hand more methods here. You have the Laplace transform, series solutions, undetermined coefficients, reduction of order, or variation of parameters. Unfortunately I can’t really give you any guidance on this other than take the path of least resistance. My rule of thumb for a series solution is that you have just non-negative integer powers of $x$ multiplied by your respective derivatives and NO singularities anywhere (within reason). I use the Laplace Transform whenever I can because it does all of the math for me at once. I use variations of parameters when I am given a solution but need another to create a fundamental set. I use undetermined coefficients when I just don’t know what to do (undetermined coefficients—in principle—always works, but the algebra is non-trivial; so unless your algebra is 100% sound, be careful). I use reduction of order when I think the solution will give me a lower order equation (don’t be afraid to use reduction of order multiple times on the same problem if it makes your life “simpler” (whatever that means)).

If the answer is just more than one equation, I try to make it a system of equations. I feel that this is the most efficient way to deal with simultaneous equations unless you want to waste you life creating a single equation from the set of equations and then risk getting stuck.
Solutions

1. \[ y = \frac{x \sin^3(x)}{3} + \frac{\cos(x)}{3} - \frac{\cos^3(x)}{3} + c \]

2. Since \( y'_h + p(t)y_h = 0 \) and \( y'_p + p(t)y_p = g(t) \), we have \((y_h + y_p)' + p(t)(y_h + y_p) = 0 + g(t) = g(t)\).

3. The solution is \( y = \frac{y_0}{1 - y_0 t^2} \) for all possible \( y_0 \) (this does not say that \( y_0 \) can be anything). If \( y_0 = 0 \), then \( y = 0 \) and is valid for all \( t \in (-\infty, \infty) \). If \( y_0 \neq 0 \), then \( y_0 > 0 \) (why?) and \( t \in \left(-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}}\right) \). (Why can't \( t \in \mathbb{R} \setminus \left[-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}}\right] \)?)

4. I couldn’t do this easily. Do it yourself. Go use Wolfram Alpha (which I couldn’t do) or do it through qualitative analysis (which is the way I did it).

5. There are eight cases
   
   (i) If \( p(t) = f(t) = g(t) = 0 \), then \( y \) = anything is a solution.
   
   (ii) If \( p(t) = f(t) = 0 \) and \( g(t) \neq 0 \), then there is no solution.
   
   (iii) If \( p(t) = g(t) = 0 \) and \( f(t) \neq 0 \), then \( y = 0 \).
   
   (iv) If \( p(t) = 0 \) and \( f(t), g(t) \neq 0 \), then \( y = \frac{g(t)}{f(t)} \).
   
   (v) If \( p(t) \neq 0 \) and \( f(t) = g(t) = 0 \), then \( y = c \) for some \( c \in \mathbb{C} \).
   
   (vi) If \( p(t) \neq 0 \) and \( f(t) = 0 \) and \( g(t) \neq 0 \), then \( y = c + \int \frac{g(t)}{p(t)} \) for some \( c \in \mathbb{C} \).
   
   (vii) If \( p(t) \neq 0 \), \( f(t) \neq 0 \) and \( g(t) = 0 \), then \( y = Ce^{-\int \frac{f(t)}{p(t)}} \) for some \( C \in \mathbb{C} \).
   
   (viii) If \( p(t), f(t), g(t) \neq 0 \), then \( y = Ce^{-\int \frac{f(t)}{p(t)}} + e^{-\int \frac{f(t)}{p(t)}} \int \frac{g(t)e^{\int \frac{f(t)}{p(t)}}}{p(t)} \) for some \( C \in \mathbb{C} \).

6. There are four cases.
   
   (i) If \( k = \omega = 0 \), then \( u = c \) for \( c \in \mathbb{C} \).
   
   (ii) If \( k = 0 \) and \( \omega \neq 0 \), then \( u = c \) for \( c \in \mathbb{C} \).
   
   (iii) If \( k \neq 0 \) and \( \omega = 0 \), then \( u = (T_0 + T_1) + Ce^{-kt} \) for \( C \in \mathbb{C} \).
   
   (iv) If \( k \neq 0 \) and \( \omega \neq 0 \), then \( u = T_0 + \frac{T_1 k \omega \sin(\omega t)}{w^2 + k^2} + \frac{T_1 k^2 \cos(\omega t)}{w^2 + k^2} + Ce^{-kt} \) for some \( C \in \mathbb{C} \).

7. (a) \( y = \frac{c}{\sin(x)} \) for some \( c \in \mathbb{C} \).
   
   (b) Write \( \frac{dy}{dx} = -\frac{stuffed}{morestuffed} \), then write \((\text{quantity})dx + (\text{quantity})dy = 0 \). Solution is \( x^3y - 3x^2 + y^3 = c \) for some \( c \in \mathbb{C} \).
   
   (c) Write \( y' + (\text{stuffed})y = (\text{otherstuffed}) \) and use integrating factor. \( y = \frac{(2 - x)(x + 1)}{2} + \frac{c(x + 1)}{x} \) for some \( c \in \mathbb{C} \)
   
   (d) Write \( y' + (\text{stuffed})y = (\text{otherstuffed}) \) and use integrating factor. \( y = \frac{c}{x^{3/2}} - \frac{1}{3} \) for some \( c \in \mathbb{C} \).
8. Write \( dx/dy \) instead of \( dy/dx \). Depending on how you integrate, you get \( 8x \sin(y) = 4 \sin^2(y) + 3 \) or \( 8x \sin(y) = 5 - 2 \cos(2y) \). These are related by the trig identity \( \sin^2(x) = \frac{1 - \cos(2x)}{2} \).

An alternate solution from the beginning is to multiply by \( \cos(y) \) and make an observation.

9. Start by differentiating both sides. After solving you should get \( y = 1 + Ce^{-x^2} \) for some \( C \in \mathbb{C} \).

10. I couldn’t do this easily. Do it yourself. Go use Wolfram Alpha (which I didn’t do) or do it through qualitative analysis (which is the way I did it).

11. (a) \( y' + y = 3 \)
(b) \( y' - y = -2 \)
(c) \( y' + y = t - 1 \)
(d) \( y' - y = 2t - 1 \)

12. \( y = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}; \lim_{t \to \infty} y(t) = M \)

13. There is no solution with the given initial value point because \( x = 2 \) isn’t even in the domain.

If your initial condition \( y(x) = y_0 \) was inside \((0, 1)\) or \((1, \sqrt{2})\) then there would be a solution.

14. If \( S \) is the amount of salt at any time \( t \), then the amount of salt in the tank at any time is given by \( \frac{dS}{dt} = 30 - \frac{4S}{t + 50} \). The concentration (in pounds per gallon) at the point of over flow is \( 14.99995 \approx 3 \).

15. You will be able to give your child \$66182.67 to put toward college.

16. The reproduction rate is \( \ln(3) \) and the initial population is \( \left[ \frac{1000}{\sqrt{3}} \right] \).

17. \( y = \frac{(t-1)e^t+1}{t^2} \)

18. A particular solution to the equation is \(-3t - \frac{1}{2} \sin(t)\). This can be done with variation of parameters or undetermined coefficients, or the Laplace transform.

19. A possible wronskian is \( \frac{1}{t^2} \).

20. (a) For the system to be critically damped, \( \gamma \) must satisfy \( \gamma^2 - 4(2)(16) = 0 \). Recall the discriminant tells you whether the system is critically damped \( (\gamma^2 - 4mk = 0) \), over-damped \( (\gamma^2 - 4mk \geq 0) \), or under-damped \( (\gamma^2 - 4mk \leq 0) \).

(b) The equation is \( 2u'' + 48u' + 16y = 0 \) subject to \( u(0) = .5 \) and \( u'(0) = 0 \).
(c) The \( \lim_{t \to \infty} y(t) = 0 \).

Every question on this review was taken from one of the following books, was made up by some Math Lab TA, or came directly off an old exam: Differential Equations (Third Edition) by Kibbey and Reddick, Elementary Differential Equations with Linear Algebra by Finney and Ostberg, Elementary Differential Equations by Boyce and DiPrima.