Math 112, Winter 2011, Exam 2, SOLUTIONS

Multiple Choice.

1. Suppose \( F(x) = f(g(x)) \), where \( f(2) = 2 \), \( g(2) = 5 \), \( f'(2) = 4 \), \( f'(5) = 8 \) and \( g'(2) = 3 \). Find \( F'(2) \).

   a) 0          b) 12          c) 8          
   d) 18         e) 24         f) None of the above. 

   Answer: e

2. Find \( \frac{dy}{dx} \) for \( x^3 + y^2 = 2x \).

   a) \( \frac{2 + 3x^2}{2y} \)       b) \( \frac{2 + 3x^2}{-2y} \)       c) \( \frac{2 - 3x^2}{-2y} \)       
   d) \( \frac{2 - 3x^2}{2y} \)   e) None of the above 

   Answer: d

3. Evaluate \( \lim_{\theta \to 0} \frac{\tan(\theta)}{\theta} \)

   a) 0                  b) 1                  c) \( \tan(\theta) \)                  
   d) \( \infty \)       e) Does not exist       f) None of the above. 

   Answer: b

4. The position of a particle at time \( t \) is described by the equation \( s(t) = t^3 - 3t \), where \( s \) is in meters and \( t \) is in seconds. Find the velocity after 2 seconds.

   a) 2 m/s       b) 12 m/s       c) 9 m/s       
   d) 3 m/s       e) 18 m/s       f) 1 m/s       
   g) 0 m/s

   Answer: c

5. If each side of a square is increasing at a rate of 5 m/s, how fast is the area of the square increasing when a side is 3 m?

   a) 10 m^2/s     b) 30 m^2/s     c) 15 m^2/s     
   d) 75 m^2/s     e) 3 m^2/s      f) 45 m^2/s      
   g) Impossible to determine.   h) None of the above. 

   Answer: b
6. If \( f(x) = \log_5(x) \), evaluate \( f'(1) \).

   a) Does not exist.  
   b) 1  
   c) \( \ln(5) \)  
   d) 0  
   e) \( \frac{1}{5} \)  
   f) \( \frac{1}{\ln(5)} \)  
   g) None of the above.

   Answer: f

7. If \( f(x) = \sin(x) \), find the 33rd derivative of \( f(x) \).

   a) \( \sin(x) \)  
   b) \( \cos(x) \)  
   c) \( \sin^{33}(x) \)  
   d) \( -\cos(x) \)  
   e) \( -\sin(x) \)  
   f) \( \tan(x) \)  

   Answer: b

8. Find the derivative of \( f(x) = 3 \tanh(x) \).

   a) \( \frac{3 \cosh(x)}{\sinh(x)} \)  
   b) \( -\frac{3 \cosh(x)}{\sinh(x)} \)  
   c) \( -3 \text{sech}(x) \tanh(x) \)  
   d) \( 3 \text{sech}^2(x) \)  
   e) \( -3 \text{csch}^2(x) \)  
   f) None of the above.

   Answer: d

9. What is the derivative of \( \tan^{-1}(e^x) \)?

   a) \( -\csc^2(e^x)e^x \)  
   b) \( \sec^2(e^x)e^x \)  
   c) \( -(\tan(e^x))^{-2}e^x \)  
   d) \( \frac{e^x}{1 + x^2} \)  
   e) \( \frac{1}{\cos(1 + x^2)} \)  
   f) \( \frac{e^x}{1 + e^{2x}} \)  

   Answer: f

10. Suppose calculus-tonium has a half-life of 5 years and we start with a 250 gram sample. If \( A(t) = 250e^{kt} \) is the function that gives the remaining amount of mass, with time measured in years, then what is \( k \)?

    a) \( k = 50 \)  
    b) \( k = \frac{1}{5} \)  
    c) \( k = \ln\left(\frac{1}{5}\right) \)  
    d) \( k = \frac{3}{4} \)  
    e) \( k = \frac{\ln\left(\frac{1}{5}\right)}{5} \)  
    f) \( k = 5 \)  
    g) None of the above

    Answer: e
11. (8 points) Use a linear approximation to estimate $\sqrt{65}$.

**Solution.**

$f(x) = x^{\frac{1}{2}}, \ a = 64$

$f'(x) = \frac{1}{2x^{\frac{1}{2}}}$

$L(x) = f(a) + f'(a)(x - a) = 8 + \frac{1}{16}(x - 64)$

$\sqrt{65} = f(65) \approx L(65) = 8 + \frac{1}{16}(65 - 64) = 8 + \frac{1}{16}$

12. (8 points) If the area of a circle is increasing at a rate of 2 $\text{cm}^2/\text{s}$, at what rate is the radius increasing when the diameter is 4 cm?

**Solution.**

$A(t) = \pi(r(t))^2$

$A'(t) = 2\pi r(t)r'(t)$

Letting $t_0$ be the instant at which we are evaluating the rate of change, we have $r(t_0) = 2$, and

$A'(t_0) = 2\pi r(t_0)r'(t_0)$, and so

$2 = 2\pi 2r'(t_0)$, thus

$r'(t_0) = \frac{1}{2\pi} \frac{\text{cm}}{\text{s}}$

13. (6 points) Find the equation of the tangent line to the curve

$y = \frac{x - 2}{x + 2}$

at the point $(−1, −3)$.

**Solution.**

Letting $f(x) = \frac{x - 2}{x + 2}$, we have $f'(x) = \frac{x + 2 - (x - 2)}{(x + 2)^2} = \frac{4}{(x + 2)^2}$

Thus $m = f'(-1) = \frac{4}{((-1) + 2)^2} = 4$, so

$y - (-3) = 4(x - (-1))$

or $y = 4x + 1$

14. (6 points) Compute the derivative of $\arccos(x)$ using implicit differentiation.

$y = \arccos(x) \Rightarrow \cos(y) = x$

**Solution.**

Differentiating,

$-\sin(y)y' = 1$, so

$y' = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1 - \cos^2(y)}} = -\frac{1}{\sqrt{1 - x^2}}$

(One can draw a triangle to directly conclude $y' = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1 - x^2}}$)
15. (8 points) If $3x^2 + 17\sin(y) = 3 + xy$, compute $\frac{dy}{dx}$ at the point (1, 0).

**Solution.**

Differentiating implicitly, we get 

$$6x + 17\cos(y)y' = xy' + y$$

and substituting the point (1, 0) we get 

$$6 + 17\cos(0)y' = 1y' + 0,$$

and solving for $y' = \frac{dy}{dx}$ gives 

$$y' = \frac{3}{8}$$

16. (8 points) Compute the derivatives of the following functions.

(a) $f(x) = 2x^2 - x + 1$

**Solution.**

$$f'(x) = \ln(2)2x^2 - x + 1(2x - 1)$$

(b) $g(x) = \ln(\sinh(x^2 + 1))$

**Solution.**

$$g'(x) = \frac{\cosh(x^2 + 1)(2x)}{\sinh(x^2 + 1)} = 2x \text{ coth}(x^2 + 1)$$

17. (8 points) Compute the derivative of $h(x) = x^x$. (Hint: Use logarithmic differentiation.)

**Solution.**

$$y = x^x, \text{ so } \ln(y) = \ln(x^x) = x \ln(x)$$

Differentiating both sides gives 

$$\frac{y'}{y} = x \frac{1}{x} + \ln(x) = 1 + \ln(x)$$

and multiplying both sides by $y$ gives 

$$h'(x) = y' = y(1 + \ln(x)) = x^x(1 + \ln(x))$$

18. (8 points) Suppose that the world population is 1500 million people in the year 1900 and 1600 million people in 1910. Give an exponential function to model world population, letting $t$ be measured in years since 1900 (so $t = 0$ is the year 1900), and letting population be measured in millions of people.

**Solution.**

$$P(t) = P(0)e^{kt} = 1500e^{kt}$$

We solve for $k$: $1600 = P(10) = 1500e^{k10}$, which means:

$$\frac{16}{15} = e^{k10}, \text{ or } \ln\left(\frac{16}{15}\right) = 10k, \text{ or } \frac{1}{10} \ln\left(\frac{16}{15}\right) = k,$$

and so we may write 

$$P(t) = 1500e^{\frac{1}{10}\ln(16/15)t}$$

or simplifying,

$$P(t) = 1500\left(\frac{1600}{1500}\right)^{t/10}$$

END OF EXAM