OBJECTIVES FOR MATH 290

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1. Chapter 1: Sets

1.1. Describing sets.

(1) Understand what a set is (and is not).
(2) Know the notations for writing a set, and for elements of a set.
(3) Know the meaning and notations for the empty set ($\emptyset$ or $\{\}$).
(4) The following is an example of a set, written using two different notations:

$$S = \{2, 4, 6, \ldots, 18\}$$

and

$$S = \{2x : x \in \mathbb{N} \text{ and } x < 10\}.$$

Be familiar with both types of notation. If some set is expressed using either of these notation types, be able to identify precisely what the elements of the set are, and be able to rewrite the same set using the other notation type.

(5) Know the meaning and notation of cardinality of a set.

(6) Know what these particular sets are: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, $\mathbb{R}$, and $\mathbb{C}$.

1.2. Subsets.

(1) Know the meaning and notation for subsets and proper subsets.
(2) Be able to distinguish between elements of a set and subsets of a set, and use the correct notation for each.
(3) Understand what “the universal set $U$” means.
(4) Know the notations for intervals of all types.
(5) Given a set, be able to find its power set. Know the notation for power set.

1.3. Set operations.

(1) Know the meaning and notation for: Union, intersection, set difference, and complement. Be able to calculate these for given sets.
(2) Be able to draw Venn diagrams to depict the above operations on sets.
(3) Know the term “disjoint.”
1.4. **Indexed collections of sets.**
   (1) Understand the notation and meaning of indexed collections of sets, and their intersection and union.
   (2) Be able to figure out an infinite intersection and infinite union of sets.

1.5. **Partitions of sets.**
   (1) Know the definition of a partition of a set. Be able to create a partition and to recognize whether something is a partition of a given set or not.

1.6. **Cartesian products of sets.**
   (1) Know what an ordered pair is, and how it differs from a set with two elements.
   (2) Know the correct notation for Cartesian products, and how to form them.

2. **Logic**

2.1. **Statements.**
   (1) Know the difference between a statement and an open sentence with domain $S$.
   (2) Be able to build truth tables for statements and open sentences.

2.2. **Negation of a statement.**
   (1) Understand what it means to form the negation of a given statement.
   (2) Be able to form the negation of statements and open sentences. As you go through the later sections in this chapter, learn to form negations of statements containing “and,” “or,” implication, and quantifiers.

2.3. **Disjunction and conjunction (“or” and “and”).**
   (1) Know the symbols, meanings, and truth tables for disjunction (“$\lor$”) and conjunction (“$\land$”).

2.4. **Implication.**
   (1) Know the symbol, meaning, and truth table for implication.

2.5. **More on implications.**
   (1) Know the terms “hypothesis” and “conclusion.”
   (2) Be able to determine the truth value of a statement involving implication. Become familiar with this.
   (3) An implication may be expressed in words without “if” and “then,” or in words with “if” and “then,” or in symbols with the implication symbol $\Rightarrow$. Be able to convert any of these into any other.
   (4) Learn from the examples in the section.
2.6. **Biconditional.**

(1) Understand the meaning of the four equivalent statements:
   
   (a) The Biconditional $P \iff Q$
   
   (b) $P$ if and only if $Q$
   
   (c) $P$ is equivalent to $Q$
   
   (d) $P$ is a necessary and sufficient condition for $Q$.

2.7. **Tautologies and contradictions.**

(1) Know the meanings of the terms “logical connectives” and “compound statement.”

(2) Know what it means for a compound statement to be a tautology (all entries in the truth table below the compound statement come out “T.”) The simplest tautology is “$P$ or not $P$.”

(3) Know what it means for a compound statement to be a contradiction (all entries in the truth table below the compound statement come out “F.”) The simplest contradiction is “$P$ and not $P$.”

(4) Be able to show that a given compound statement is (or is not) a tautology or a contradiction.

2.8. **Logical equivalence.** (See 2.9 below)

2.9. **Some fundamental properties of logical equivalence.**

(1) There are no objectives specifically listed under sections 2.8 or 2.9. In connection with other sections, you should know the term “logically equivalent,” and be able to use truth tables to show that two statements are logically equivalent.

2.10. **Quantified statements.**

(1) Understand the meaning and notation for the universal quantifier $\forall$.

(2) Understand the meaning and notation for the existential quantifier $\exists$.

(3) Be able to evaluate whether a given quantified statement is true or false.

(4) Be able to negate a quantified statement.

(5) Understand why order matters; for example, $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y)$ is different from $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x < y)$.

2.11. **Characterizations of statements.**

— No objectives —
2.12. **Typesetting in LaTeX.** This is not in the textbook. You will have a handout to work through. LaTeX is the preferred system for typesetting mathematics, and you will need to use it to type some of your homework assignments. Learning LaTeX is one of the objectives of the course, but we will not include detailed learning objectives right here.

3. **Direct proof and proof by contrapositive**

3.1. **Trivial and vacuous proofs, and truth table depictions of a proof.**

1. Know what a trivial proof is: When $Q(x)$ is true for all $x$, then a proof that $P(x) \Rightarrow Q(x)$ doesn’t have to even mention $P(x)$. Such a proof is called a trivial proof.
2. Know what a vacuous proof is: When $P(x)$ is false for all $x$, then a proof that $P(x) \Rightarrow Q(x)$ doesn’t have to mention $Q(x)$, in which case it is called a vacuous proof.
3. Understand a truth table depiction of a proof that $(\forall x)(P(x) \Rightarrow Q(x))$. The idea is that the only thing that could invalidate $P(x) \Rightarrow Q(x)$ is if there is a TF pair, that is, a T under $P(x)$ and an F under $Q(x)$ for the same $x$. So in any valid proof, our job is to show there are no TF pairs. This idea is further developed in the sections below.

3.2. **Direct proofs.**

1. Know that in a direct proof of $P(x) \Rightarrow Q(x)$ we start by assuming $P(x)$ is true, and work until we can show that $Q(x)$ is true.
2. Understand the truth table depiction of a direct proof: “Gather up all the T’s under $P(x)$ and show that none of them have F’s beside them under $Q(x)$.”
3. Note: In the truth table depiction of a vacuous proof, you go to gather up all the T’s under $Q(x)$ and there aren’t any, so you don’t have to worry about showing there couldn’t be any F’s under $Q(x)$ that go with them.
4. Know the definition of an even integer and the definition of an odd integer.
5. Be able to do proofs like those in the exercises concerning even and odd numbers.

3.3. **Proof by contrapositive.**

1. Know that a proof by contrapositive of $P(x) \Rightarrow Q(x)$ starts by assuming $Q(x)$ is false and works to show $P(x)$ is false.
2. The truth table depiction of a proof by contrapositive is: “Gather up all the F’s under $Q(x)$ and show that none of them have T’s beside them under $P(x)$.
3. Note: In the truth table depiction of a trivial proof, you go to gather up all the F’s under $Q(x)$ and there aren’t any, so you don’t have to worry about showing there couldn’t be any T’s under $P(x)$ that go with them.
3.4. **Proof by cases.**

(1) Understand when to break into cases:
   - In any proof, you can break into cases at any time, as long as the cases you list cover all possibilities.
   - As a practical matter, you often break into cases when you can see two or more possibilities that may be true, and you want to pin down the possibilities one at a time.

In a proof about sets, for example, it may be helpful to know whether a given element \( x \) is in \( A \) or not. If you don’t know, then you can say
   
   “Case 1: \( x \in A \) ...
   
   Case 2: \( x \notin A \).”

(2) Understand that in a proof by cases, you must reach the desired result at the end of each case.

(3) Know the meaning of “WLOG,” (without loss of generality). Know when you are allowed to use it to shorten a proof.

3.5. **Proof evaluations.**

(1) Practice proof evaluations.

4. **More on direct proof and proof by contrapositive**

4.1. **Proofs involving divisibility of integers.**

(1) Know the meaning and notation for “\( a \) divides \( b \),” which is “\( a|b \).”

   Note 1: Do not confuse the statement \( a|b \) with the fraction \( a/b \) or \( b/a \). For example, the expression “3|12” does **not** represent the number 4 or the number 1/4; instead it represents the (true) statement that 3 divides evenly into 12.

   Note 2: Notice that for \( a \neq 0 \), the definition of \( a|b \) is that there exists an integer \( c \) such that \( ac = b \). This statement uses multiplication, which is preferred over the alternative definition “\( b \) divided by \( a \) is an integer.”

(2) Be able to write divisibility proofs like any of those in this section.

4.2. **Proofs involving congruence of integers.**

(1) Know the definition and notation for “\( a \) is congruent to \( b \) modulo \( n \),”

(2) Know how to use results 4.9 through 4.11 in the book.
(3) Know the possibilities for congruence mod 3: For any integer \( n \), it will be true that \( n \equiv 0 \pmod{3} \) or \( n \equiv 1 \pmod{3} \) or \( n \equiv 2 \pmod{3} \). Similarly, for any integer \( k \geq 2 \), any integer \( n \) will be equivalent to one of: 0, 1, 2, \ldots, \( k - 1 \) (mod \( k \)).

(4) Be able to write congruence proofs like any of those in this section.

Note: Congruence modulo \( n \) is a very rich topic and is one of the foundational concepts of number theory. We will revisit this topic in chapter 11.

4.3. Proofs involving real numbers.

(1) Many times a proof of an inequality can be done by finding a perfect square. Example:
If \( a > 0 \) then \( a + 1/a \geq 2 \).
Proof: Let \( a > 0 \). Then

\[
a + 1/a - 2 = 1/a(a^2 - 2a + 1) = 1/a(a - 1/a)^2
\]

which is nonnegative because it is a positive number times a perfect square, so

\[
a + 1/a - 2 \geq 0
\]

\[
a + 1/a \geq 2.
\]

(2) Know the definition of absolute value, and know the triangle inequality.

(3) Know that

\[
|x| \leq y
\]

if and only if

\[
-y \leq x \leq y.
\]

Be able to recognize situations where this will help in a proof.

4.4. Proofs involving sets.

(1) Know the meaning and notation for set intersection, union, difference, and complement.

(2) Be able to do “element-wise” set proofs. For example, to prove that a set \( A \) is a subset of another set \( B \), start with an arbitrary \( x \in A \) and work till you show that \( x \in B \). To prove that \( A = B \), first take an arbitrary \( x \in A \) and work to show \( x \in B \). Second, start with an arbitrary \( y \in B \) and work until you show that \( y \in A \).

(3) Note: Element-wise proofs are not always the best way; see section 4.5 below.
4.5. **Fundamental properties of set operations.**

(1) Know all the laws in theorem 4.21. A test question might say “List both distributive laws of sets.” Or, “List both of DeMorgan’s laws.”

(2) Be able to do proofs like exercise 4.39 using the laws in theorem 4.21. This method is sometimes a good alternative for element-wise proofs.

4.6. **Proofs involving cartesian products of sets.**

(1) Know the definition and notation of the Cartesian product of two sets $A$ and $B$. Be able to do proofs about Cartesian products.

(2) Be careful of a pitfall: To take an arbitrary element of $A \times A$, don’t just take $(x, x)$ where $x \in A$. This would not be capable of representing all elements of $A \times A$. Instead, take an arbitrary element $(x, y)$, where $x \in A$ and $y \in A$.

5. **Existence, and Proof by Contradiction**

5.1. **Counterexamples.**

(1) Understand what a counterexample is, for an if-then statement or a statement of the form “$\forall x \in S, R(x)$.”

(2) Understand the role of a counterexample in disproving a statement.

5.2. **Proof by contradiction.**

(1) Understand how to set up a proof by contradiction. For example, a proof by contradiction for the theorem

$$\text{If } x^3 < 0 \text{ then } x < 0$$

begins with the statement

“Assume that $x^3 < 0$ and, by way of contradiction, assume that $x \geq 0$,”

or, more briefly,

“Assume that $x^3 < 0$ and that $x \geq 0$.”

**Note: Avoid this pitfall:** Do not use the word “if” or the word “then” at the beginning of a proof by contradiction. In the example above it would be incorrect logic to begin with the words “Assume that if $x^3 < 0$ then $x \geq 0$.” If a proof began in this way and were perfect from that point on, it would at best succeed in proving a different theorem than the one that was asked for.

Sometimes students avoid the pitfall only halfway by writing “Assume that $x^3 < 0$, then $x \geq 0$. This is ambiguous and suggests that the writer only halfway understands. Instead, to avoid this pitfall you need to first fully understand why and how a proof by contradiction works, and then not use the word “if” or the word “then” in the opening line of the proof.
(2) Be able to complete proofs by contradiction like those in the section and the homework.

(3) Understand the advantages and disadvantages of a proof by contradiction. The advantage is that you get as many hypotheses as possible — as much as possible “on the table” to work with. This may be very helpful.

A second advantage is that negative conclusions get transformed (by negation) into positive hypotheses. Notice, for example, the proofs in the book about irrational numbers.

A disadvantage of proof by contradiction may be that you are not sure in what direction to go to reach the contradiction.

Another disadvantage is that proof by contradiction is viewed by some as less desirable. You certainly want to avoid a proof by contradiction that is really a direct proof in disguise. A simple (silly) example is the following:

Theorem: If $x = 3$ then $2x = 6$.

Proof by contradiction. Assume that $x = 3$ and, by way of contradiction, assume that $2x \neq 6$. Since $x = 3$ we can multiply both sides by 2 to get $2x = 6$. But we assumed that $2x \neq 6$, which is a contradiction. So the original theorem is true.

A proof like that is downright impolite.

5.3. **A review of three proof techniques.**

(1) Take this opportunity to review direct proof, proof by contrapositive, and proof by contradiction, by studying the examples in this section.

5.4. **Existence proofs.**

(1) Understand that an existence proof sometimes works by finding a specific object that satisfies the given statement, and sometimes does not.

(2) Be able to write an existence proof using the intermediate value theorem, such as in the book’s Result 5.22.

5.5. **Disproving existence statements.**

(1) Understand that to disprove an existence statement is really to prove a related “for all” statement.

6. **Mathematical Induction**

6.1. **The principle of mathematical induction.**

(1) Know the well-ordering principle.

(2) Be able to write a proof by induction for a formula, an inequality, and a divisibility statement (like numbers 11, 14, and 21 in the homework.)
6.2. **A more general principle of mathematical induction.**
   (1) Be able to do proofs by induction when the base case is not \( n = 1 \), as in section 6.2.

6.3. **Proof by minimum counterexample.**
   (1) Be able to explain the idea of proof by minimum counterexample.

6.4. **The strong principle of mathematical induction.** No objectives

7. **Note:** We will not have any learning objectives specific to chapter 7.

8. **Equivalence Relations**

8.1. **Relations.**
   (1) Be able to define a relation from \( A \) to \( B \), and a relation on \( A \).
   (2) Be able to find the domain and range of a given relation, as well as the relation’s inverse.

8.2. **Properties of relations.**
   (1) Know the definitions of reflexive, symmetric and transitive.
   (2) For each of these three properties, be able to check whether a given relation has that property or not.

8.3. **Equivalence relations.**
   (1) Know the definition of an equivalence relation.
   (2) Be able to test a relation to see whether it is an equivalence relation, and if so, prove that it is.

8.4. **Properties of equivalence classes.**
   (1) Know what an equivalence class is.
   (2) Know what a partition is.
   (3) Given an equivalence relation, be able to write down the partition that goes with it, and given a partition of a set, be able to write down the equivalence relation that goes with it.
   (4) Be able to prove Theorem 8.3 (an equivalence relation always gives you a partition).
   (5) Be able to prove Theorem 8.4 (a partition always gives you an equivalence relation).
8.5. **Congruence modulo** \( n \).

(1) Be able to prove Theorem 8.6 (congruence modulo \( n \) is an equivalence relation on \( \mathbb{Z} \)).

8.6. **The integers modulo** \( n \).

(1) Know what it means to add two equivalence classes in the integers modulo \( n \).

(2) Understand the term “well-defined.”

(3) Be able to prove Theorem 8.9 (addition in \( \mathbb{Z}_n \), \( n \geq 2 \), is well-defined).

9. **Functions**

9.1. **The definition of a function.**

(1) Know the definition of a function \( f : A \to B \). Note that

(a) Every \( x \) in \( A \) must get mapped to some point in \( B \). That is,

\[(\forall x \in A) (\exists y \in B)(f(x) = y)\]

And (b) No \( x \) can get mapped to two places. That is,

\[(\forall x \in A)(\forall y_1, y_2 \in B)([f(x) = y_1 \text{ and } f(x) = y_2] \Rightarrow y_1 = y_2)\]

(2) Know the terms domain, codomain, range, image, and inverse image.

(3) Know Dirichlet’s function (page 219).

9.2. **The set of all functions from** \( A \) **to** \( B \).

(1) Know the notation \( B^A \), where \( A \) and \( B \) are sets.

(2) Recall and understand the formula

\[|B^A| = |B|^{|A|}\]

9.3. **One-to-one and onto functions.**

(1) Know the definition of a function \( f : A \to B \) that is “onto:”

\[(\forall y \in B)(\exists x \in A)(f(x) = y)\]

Notice how this is similar to the first requirement in objective 1 of section 9.1. Don’t confuse one of these for the other.

(2) Know the definition of a function \( f : A \to B \) that is “one-to-one.”

\[(\forall x_1, x_2 \in A)(f(x_1) = f(x_2) \Rightarrow x_1 = x_2)\]

(3) Know the alternative definition for one-to-one, that involves a contrapositive of an implication in the first definition:

\[(\forall x_1, x_2 \in A)(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))\]
Don’t fall into the trap of confusing the two definitions for one-to-one and making a statement like “$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$”, which is a statement that is true for ALL functions, not just for one-to-one functions. 

(4) Be able to write proofs about one-to-one (injective), onto (surjective) and compositions of functions. These can be found in the text and the exercises for the section.

9.4. Bijective functions.

(1) Know the definition of a bijective function.
(2) Given a function that is bijective, be able to prove that it is bijective.
(3) Be able to prove Theorem 9.7.

9.5. Composition of functions.

(1) Know the definition of composition of two functions.
(2) Get good at proofs that involve composition and the concepts of one-to-one and onto.

9.6. Inverse functions.

(1) Know the definition of the inverse of a function.
(2) Given a simple function, be able to find its inverse.
(3) Be able to prove Theorem 9.16.

9.7. Permutations.

(1) Know the definition and notation for a permutation.
(2) Be able to calculate inverses and compositions of permutations.
(3) Understand that permutations do not always commute. That is, $\alpha \circ \beta \neq \beta \circ \alpha$ for some permutations $\alpha$ and $\beta$. 