1.341 Objectives, Chapter 1

1.1 Discussion: The Irrationality of $\sqrt{2}$

Schedule: Lecture

1. Be able to prove that there is no rational number whose square is 2.

2. Know what the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ represent. Name something you can do in each one that you cannot do in the previous one. Which ones are “fields”?

1.2 Some Preliminaries

1. Understand union, intersection, and complements of sets.

2. Understand subsets and set equality.

3. Understand the meaning of the union or intersection of an infinite collection of sets.

4. Know what Dirichlet’s function is.

5. Know the rules about addition, subtraction (see Exercise 1.2.5), multiplication and division of absolute values: The triangle inequality, the reverse triangle inequality, and the equations

$$|ab| = |a||b| \quad \text{and} \quad \left| \frac{a}{b} \right| = \frac{|a|}{|b|}.$$ 

6. Know the meanings of proof by contradiction and of “if and only if.”

7. Understand what is meant by the quantifiers “for all” and “there exists.”

8. Be able to do a proof by induction.

9. Be able to form the negation of a claim. (See Exercise 1.2.8)
1.3 The axiom of completeness

1. What property does \( \mathbb{R} \) have that \( \mathbb{Q} \) does not have? Be able to illustrate with an example.

2. Be able to write down the axiom of completeness. Don’t leave out the slippery word “nonempty.”

3. Be able to define “bounded above” and “least upper bound.” Know the notations “sup \( S \)” and “inf \( S \)”.

4. Understand the difference between the maximum element of \( S \) and sup \( S \).

5. Know Lemma 1.3.7. It may be described as the “left-right step.” The slippery piece is the right step: “there exists \( a \in A \)...”

6. Be able to prove statements about the supremum of a set using the ideas of Lemma 1.3.7; for example, Exercises 1.3.4 and 1.3.5.

1.4 Consequences of Completeness

1. Be able to prove the Nested Interval Property.

2. Be able to define density and prove that \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

3. Be able to prove that there is a real number whose square is 2.

4. Be able to prove that \( \mathbb{Q} \) is countable.

5. Be able to prove that \( \mathbb{R} \) is uncountable.


7. Know what the Schröder-Bernstein Theorem says (see Exercise 1.4.13).

1.5 Cantor’s Theorem and 1.6 Epilogue

1. Be able to prove Cantor’s theorem.

2. Know a mathematical statement that is undecidable.