1. Let $A$ be an invertible $n \times n$ matrix with integer entries. Prove that $A^{-1}$ has integer entries if and only if $\det A = \pm 1$.

2. (a) Find a $3 \times 3$ matrix $A$ that has eigenvalues 0, 1, $-1$ with corresponding eigenvectors $(0, 1, -1)^T, (1, -1, 1)^T, (0, 1, 1)^T$.

(b) Is this answer unique? Explain.

3. Prove that if the vectors $x, y, z$ in a vector space $V$ are linearly independent, then so are the vectors $x + y, y + z, x + z$.

4. Let $G$ be a group, and let $H$ be a subgroup of $G$ of finite index $n$. Let $e$ be the identity of $G$.

   (a) Show that if $H$ is normal in $G$, then $x^n \in H$ for all $x \in G$.

   (b) Is the statement in the first part true if the hypothesis of normality is dropped? Prove or disprove.

5. How many elements of order 5 are contained in a group of order 20? Justify your answer.

6. If $\gcd(m, n) = d$, prove that the system

\[
\begin{align*}
x &\equiv a \pmod{m} \\
x &\equiv b \pmod{n}
\end{align*}
\]

has a solution if and only if $a \equiv b \pmod{d}$. If $s, t$ are solutions of this system, prove that $s \equiv t \pmod{r}$, where $r$ is the least common multiple of $m$ and $n$.

7. (a) Determine all ideals of the ring $R = \mathbb{Z}[x]/(2, x^3 + 1)$.

(b) Determine the number of elements of $R$.

(c) Write $R$ as a direct product of fields if possible.

8. Determine the number of monic irreducible cubic polynomials in $F_p[x]$, where $F_p$ denotes the field with $p$ elements.

9. Factor each of the following into irreducible factors in $\mathbb{Q}[x]$. Justify your answer.

   (a) $x^4 + 4$

   (b) $x^4 - 4x^3 + 6$

   (c) $x^3 + x + 1$

10. Give an example of an infinite field of characteristic 5.