1. Give the definitions of both nilpotent groups and solvable groups. Prove that nilpotent groups are always solvable.
2. Prove that every non-trivial ideal in a ring with 1 is contained in a maximal ideal.
3. Compute the Galois group over \( \mathbb{Q} \) for the splitting field of the polynomial \( x^4 + 4 \).
4. Find the rational canonical form and Jordan canonical form for the matrix
\[
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
5. Let \( \zeta_n \) be a primitive \( n \)th root of unity in \( \mathbb{C} \). Prove that \( \mathbb{Q}(\zeta_n)/\mathbb{Q} \) is an Abelian extension.
6. Let \( R \) be a ring with 1, and let \( A \) be a set. Prove that there is a free module \( F(A) \) on the set \( A \). In other words, there is a right \( R \)-module \( F(A) \), containing the set \( A \), satisfying the following universal property: if \( M \) is a right \( R \)-module, and \( \varphi : A \to M \) is a set map, then there is a unique right \( R \)-module homomorphism \( \Phi : F(A) \to M \) such that \( \Phi(a) = \varphi(a) \) for every \( a \in A \).
7. Let \( F \) be a field. Prove that the only ideals in \( \mathbb{M}_n(F) \) (the ring of \( n \times n \) matrices over \( F \)) are the zero ideal and the entire ring.
8. Prove Hilbert’s Basis Theorem: If \( R \) is a Noetherian ring then so is the polynomial ring \( R[x] \).
9. Classify all groups of order \( pq \) where \( p < q \) are primes.
10. Let \( G \) be a group of order 30. Prove that \( G \) has a normal subgroup isomorphic to \( \mathbb{Z}/15\mathbb{Z} \) (and in particular, is not simple).