Algebra Ph.D. Qualifying Exam, August 2011

Answer all questions. Partial credit will be given.

1. Give the definitions of both nilpotent groups and solvable groups. Prove that nilpotent groups are always solvable.
2. Prove that every non-trivial ideal in a ring with 1 is contained in a maximal ideal.
3. Compute the Galois group over $\mathbb{Q}$ for the splitting field of the polynomial $x^4 + 4$.
4. Find the rational canonical form and Jordan canonical form for the matrix $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
5. Let ζ_n be a primitive nth root of unity in $\mathbb{C}$. Prove that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is an Abelian extension.
6. Let R be a ring with 1, and let A be a set. Prove that there is a free module $F(A)$ on the set A. In other words, there is a right R-module $F(A)$, containing the set A, satisfying the following universal property: if M is a right R-module, and $\varphi : A \to M$ is a set map, then there is a unique right R-module homomorphism $\Phi : F(A) \to M$ such that $\Phi(a) = \varphi(a)$ for every $a \in A$.
7. Let F be a field. Prove that the only ideals in $\mathbb{M}_n(F)$ (the ring of $n \times n$ matrices over F) are the zero ideal and the entire ring.
8. Prove Hilbert's Basis Theorem: If R is a Noetherian ring then so is the polynomial ring $R[x]$.
9. Classify all groups of order pq where $p < q$ are primes.
10. Let G be a group of order 30. Prove that G has a normal subgroup isomorphic to $\mathbb{Z}/15\mathbb{Z}$ (and in particular, is not simple).