Algebra Ph.D. Qualifying Exam, August 2012
Answer all questions. Partial credit will be given.

1. Prove that there is no simple group of order 105.

2. Let R be a Euclidean domain. Prove that every nonzero prime ideal contains a prime element.

3. Find a primitive element for the extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$, where ω is a primitive cube-root of one. (You must prove that the element you found really is a simple generator for the extension.)

4. Find the characteristic polynomial, the minimal polynomial, and the invariant factors (over $\mathbb{Q}$) of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Use these to find the Jordan canonical and rational canonical forms.

5. Let p be an odd prime. Describe four non-isomorphic groups of order p^3, and prove they are non-isomorphic.

6. Let R and S be commutative rings with $1 \neq 0$. Prove that the ideals of $R \times S$ are precisely the sets of the form $I \times J$ where I is an ideal of R and J is an ideal of S. In particular, prove that $R \times S$ is never a field.

7. Prove that the center of a nontrivial, finite p-group is always nontrivial.

8. Prove that an infinite dimensional vector space V over $\mathbb{F}_2$ is not isomorphic to its dual $V^* = \mathrm{Hom}_{\mathbb{F}_2}(V, \mathbb{F}_2)$. (You may freely use true facts about cardinalities, if it is helpful.)

9. Let R be a commutative ring with $1 \neq 0$. Prove that if $f(x)g(x) = 0$ for some nonzero polynomials $f(x), g(x) \in R[x]$ then there exists a nonzero element $c \in R$ such that $f(x)c = 0$.

10. Prove that irreducible polynomials over $\mathbb{Q}$ are always separable. Do the same for polynomials over finite fields.