PH. D. QUALIFYING EXAM FALL 2010 - ALGEBRA

Answer all the questions. For each question give appropriate proofs.

1. Let p, q be prime numbers. Show that a group of order p^2 is abelian and that a group of order $pq, p < q$, cannot be simple.

2. (a) Let N, M be normal subgroups of the group G, where $G = MN$. Prove that

$$G/(M \cap N) \cong G/N \times G/M.$$

(b) Let $m, n \in \mathbb{N}$. Show that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$.

3. (a) List the non-isomorphic abelian groups of order 600?
(b) How many elements of order 2 does each of these groups have?

4. Let $D_8 = \langle a, b | a^2, b^2, (ab)^4 \rangle$ be the dihedral group of order 8. Let $H = \langle b \rangle$. Then D_8 acts on the right cosets of H in D_8 to give a permutation representation of D_8 as a subgroup of S_4. Let $V = \mathrm{Span}_{\mathbb{C}}(\{e_1, e_2, e_3, e_4\})$ be the corresponding $\mathbb{C}D_8$ permutation module, where D_8 permutes the indices of the basis $\{e_1, e_2, e_3, e_4\}$. Show that there are irreducible submodules W_1, W_2, W_3 of V having dimensions $1, 1, 2$ (respectively) such that $V = W_1 \oplus W_2 \oplus W_3$.

5. Let R be a principal ideal domain. Show that a non-zero element of R is a prime if and only if it is irreducible.

6. Let $\phi : V \to W$ be a linear transformation of vector spaces (not necessarily finite-dimensional). Show that the kernel ($\ker \phi$) and image ($Im(\phi)$) of ϕ are subspaces of V and W (respectively), and that

$$\dim V = \dim \ker \phi + \dim Im(\phi).$$

7. Let $F \subseteq K \subseteq L$ be fields. Prove: If the extensions L/K and K/F are algebraic, then L/F is also algebraic.

8. Let R be a ring with $1 \neq 0$. Recall that an idempotent is an $e \in R$ such that $e^2 = e$. Let e be a central idempotent. Show that Re and $R(1-e)$ are 2-sided ideals of R and that $R \cong Re \times R(1-e)$.

9. Fix $r \in \mathbb{N}$. Determine all fields $\mathbb{Q} \subset K \subset \mathbb{C}$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong C_2^r$, where C_2 is the cyclic group of order 2.

10. Determine, up to isomorphism, the Galois group of each of the following polynomials:
(1) $x^4 - 49 \in \mathbb{Q}[x]$ over $\mathbb{Q}$;
(2) $x^4 + 49 \in \mathbb{Q}[x]$ over $\mathbb{Q}$;
(3) $x^5 - 4x + 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.