PhD Algebra Qualifying Exam
August 2015

1. Let $G = S_6$, the symmetric group on six objects. For each prime dividing the order of G find a Sylow p-subgroup of G. How many Sylow p-subgroups of G are there for each p?

2. Let $\mathbb{Z}$ be the group of integers (with additive notation) and put $G = \mathbb{Z} \times \mathbb{Z}$. Let $a, b, c, d \in \mathbb{Z}$ with $ad - bc \neq 0$. Let H be the subset of all elements of the form
$$(an + bm, cn + dm), \text{ where } n, m \in \mathbb{Z}.$$
Show that H is a subgroup of G and that $|G/H| = |ad - bc|$.

3. Let $G = UT(3, \mathbb{F}_3)$, the group of invertible upper triangular 3×3 matrices with coefficients in $\mathbb{F}_3$. Find four maximal subgroups of G.

4. Determine all rings R where $\mathbb{Z} \subset R \subset \mathbb{Q}$.

5. Let $\varphi : R \to R'$ be a ring homomorphism of commutative rings, and let $\mathfrak{q} \subset R'$ be an ideal.

   (a) Prove that $\varphi^{-1}(\mathfrak{q})$ is an ideal and that it is prime if $\mathfrak{q}$ is.

   (b) Show that $\mathfrak{q}$ is prime if $\varphi^{-1}(\mathfrak{q})$ is prime and φ is surjective.

6. Let e, e' be idempotents in a commutative ring with identity R (so that $e^2 = e, (e')^2 = e'$). Show that $e'' := e + e' - ee'$ is also idempotent, and that $\langle e, e' \rangle = \langle e'' \rangle$.

7. Suppose that R is a domain, that $x \in R$ is nonzero, and that $M = R[x^{-1}]$. Suppose that M is finitely generated as a module over R. Show that $x^{-1} \in R$ and that $R = M$.

8. Prove or disprove: There exists a field F of characteristic 0, such that every quadratic polynomial over F splits, but there are irreducible degree 4 polynomials over F.

9. Assume F is a field of characteristic 0. Let $f(x) \in F[x]$ be a monic, irreducible polynomial of degree n, and let K be a splitting field for f over F. Fix distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $f(x)$ inside K.

   Let $D := \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2$ denote the discriminant of $f(x)$. Prove that the Galois group of K/F is a subgroup of A_n (after identifying the Galois group with the permutation group on the roots of f) if and only if $D \in F^2 = \{f^2 : f \in F\}$.

10. Let $A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{pmatrix}$ be matrices in $\mathbb{M}_3(\mathbb{F})$, for a field $\mathbb{F}$.

    Part (a): For which values of $c \in \mathbb{F}$ is A similar to B when $\mathbb{F} = \mathbb{Q}$?

    Part (b): For which values of $c \in \mathbb{F}$ is A similar to B when $\mathbb{F} = \mathbb{F}_3$?

    Part (c): For which values of $c \in \mathbb{F}$, for some extension field $\mathbb{F}$ of $\mathbb{Q}$, does A commute with B?