Algebra Ph.D. Qualifying Exam

August 2016

Answer all 10 questions. Your judgment as to which theorems are appropriate to use for each problem is part of what is being tested, so be sure not to use theorems which make the problems trivial.

(1) Let p be a prime number and n a nonnegative integer. Prove that every group of order p^n is solvable.

(2) Let G be a group, and let H and K be nontrivial proper normal subgroups of G, with H not isomorphic to K. Assume that all of H, K, G/H, and G/K are simple. Prove that H is isomorphic to G/K.

(3) Let P be a normal Sylow p-subgroup of a group G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

(4) Let M be an $n \times n$ matrix over $\mathbb{C}$ with finite order. Prove that M is diagonalizable.

(5) Let V be a finite-dimensional vector space over the real numbers, and let $\phi : V \to V$ be a linear transformation. If $\phi(\phi(v)) = -v$ for all $v \in V$, prove that the dimension of V is even.

(6) Let F be a finite field with p elements, and let M be a $p \times p$ matrix over F with all entries equal to 1. Find the rational canonical form of M over F.

(7) Let $i \in \mathbb{C}$ be a square root of -1. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

(8) Let $K = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt{2 + \sqrt{2}}$. Let $L/\mathbb{Q}$ be the Galois closure of $K/\mathbb{Q}$.
 (a) Determine the elements of $\mathrm{Gal}(L/\mathbb{Q})$ by describing where each element maps α.
 (b) Determine the isomorphism class of $\mathrm{Gal}(L/\mathbb{Q})$.

(9) Determine (with proof) whether the polynomial $x^5 - 6x + 1$ is solvable by radicals over $\mathbb{Q}$. If it is, determine a root in terms of radicals.

(10) Let L/K be a finite extension of finite fields (*i.e.* L and K are finite fields, and $[L : K]$ is finite). Prove that L/K is a separable extension.