Algebra Ph.D. Qualifying Exam, January 2012
Answer all questions. Partial credit will be given.

1. Prove that a free group on any generating set with more than one element is non-Abelian.

2. Prove that every commutative ring with $1 \neq 0$ has a minimal prime ideal. You may freely use the fact that such rings have maximal ideals if that is helpful.

3. Compute the Galois group over $\mathbb{Q}$ for the splitting field of the polynomial $x^4 + x^3 + x^2 + x + 1$.

4. Find the rational canonical form and Jordan canonical form for the integer matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

5. Prove that $\mathrm{Aut}(Q_8) \cong S_4$ where $Q_8 = \langle i, j, k \ : \ i^2 = j^2 = k^2 = -1, \ (-1)^2 = 1, \ ij = k = -ji, \ ki = j = -ik, \ jk = i = -kj \rangle$ is the quaternion group. (You are free to use any presentation of this group you like.)

6. Prove that a PID has the property that any ascending chain of ideals stabilizes.

7. Classify all groups of order p^2, where p is prime.

8. Let F be a field. Let V be a finite dimensional vector space. Prove that V is isomorphic to its double dual $(V^*)^*$.

9. Is the (possibly infinite) direct product of fields a PID? Justify your answer.

10. Let p be a prime. Prove that there is a field of order p^n, for each $n \geq 1$.