PH. D. QUALIFYING EXAM SPRING 2010 - ALGEBRA

Answer all the questions. For each question give appropriate proofs.

1. For a prime p let C_p denote the cyclic group of order p. How many subgroups of order p^5 does $C_p^6 = C_p \times C_p \times C_p \times C_p \times C_p \times C_p$ have? (Remember to prove your assertion.)

2. Prove the first part of Sylow's theorem: let G be a finite group of order $|G| = p^k n$, where p is a prime and $\gcd(p, n) = 1$; then G has a subgroup of order p^k.

3. Prove that for any polynomial $f \in F[x]$, where F is a field, there is a splitting field for f over F.

4. The dihedral group D_8 of order 8 acts on the corners of a square to give a permutation representation $D_8 = \langle (1, 2, 3, 4), (1, 3) \rangle$ as a subgroup of S_4. Let $V = \mathrm{Span}_{\mathbb{C}}(\{e_1, e_2, e_3, e_4\})$ be the corresponding $\mathbb{C}D_8$ permutation module, where D_8 permutes the indices of the basis $\{e_1, e_2, e_3, e_4\}$. Show that there are irreducible submodules W_1, W_2, W_3 of V having dimensions $1, 1, 2$ (respectively) such that $V = W_1 \oplus W_2 \oplus W_3$.

5. Let R be a commutative ring with 1, and let N be the set of all nilpotent elements in R. Prove that N is an ideal and that the ring R/N contains no nonzero nilpotent elements.

6. For $n \geq 1$ let J_n denote the all 1 square matrix with n rows, and let I_n be the identity matrix with n rows. Determine the Jordan canonical form of $J_n - I_n$.

7. Let $\zeta \in \mathbb{C}$ be a primitive 17th root of unity. For each element α listed below, find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
(a) $\alpha = \zeta$;
(b) $\alpha = \zeta + \zeta^{16}$;
(c) $\alpha = \zeta + \zeta^2 + \zeta^4 + \zeta^8 + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16}$.

8. Let F be a field and let $f(x) \in F[x]$. Show that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over F.

9. Determine, up to isomorphism, all groups of order 8.

10. Determine, up to isomorphism, the Galois group of each of the following:
(1) $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ over $\mathbb{Q}$;
(2) $x^6 + 1 \in \mathbb{R}[x]$ over $\mathbb{R}$;
(3) $x^5 - 12x + 1 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.