Algebra Ph.D. Qualifying Exam, Spring 2011

Answer all questions. Partial credit will be given.

1. Prove that $A_5$ is a simple group.

2. Let $R$ be a commutative ring with 1. Prove that the following properties on $R$ are equivalent:
   a. If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ is a chain of ideals in $R$, then there is an integer $n$ so that $I_m = I_n$ for all $m \geq n$. (There are no infinite increasing chains of ideals.)
   b. Every ideal in $R$ is finitely generated.

3. Compute the Galois group over $\mathbb{Q}$ for the splitting field of the minimal polynomial for $\alpha = \sqrt{-3} + \sqrt{2}$.

4. Let $V$ be a finite dimensional vector space over a field $F$. Let $\varphi : V \to V$ be a linear transformation. Prove that there exists an integer $m \geq 1$ so that $\ker(\varphi^m) \cap \text{im}(\varphi^m) = \{0\}$.

5. Suppose $M/L$ and $L/K$ are algebraic extensions of fields. Prove that $M/K$ is algebraic. (You may not assume these extensions are finite.)

6. Let $R$ be a ring with 1 and let $M$ be a Noetherian right $R$-module. Let $f : M \to M$ be a surjective $R$-module homomorphism. Prove that $f$ is also injective.

7. Let $A, B \in \mathbb{M}_n(\mathbb{Q})$ be two $n \times n$ matrices defined over $\mathbb{Q}$. If $A$ and $B$ are similar over $\mathbb{C}$, are they similar over $\mathbb{Q}$? Justify your answer.

8. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

9. Let $P$ be a normal Sylow $p$-subgroup of $G$ and let $H$ be any subgroup of $G$. Prove that $P \cap H$ is the unique Sylow $p$-subgroup of $H$.

10. Find the Jordan Canonical Form for the matrix
    
    \[
    \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    1 & 0 & 0 & -4 \\
    0 & 1 & 0 & 4 \\
    0 & 0 & 1 & 1
    \end{pmatrix}
    \]