Algebra Ph.D. Qualifying Exam, Spring 2011

Answer all questions. Partial credit will be given.

1. Prove that A_5 is a simple group.
2. Let R be a commutative ring with 1. Prove that the following properties on R are equivalent:
   a. If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is a chain of ideals in R, then there is an integer n so that $I_m = I_n$ for all $m \geq n$. (There are no infinite increasing chains of ideals.)
   b. Every ideal in R is finitely generated.
3. Compute the Galois group over $\mathbb{Q}$ for the splitting field of the minimal polynomial for $\alpha = \sqrt{-3} + \sqrt[3]{2}$.
4. Let V be a finite dimensional vector space over a field F. Let $\varphi : V \to V$ be a linear transformation. Prove that there exists an integer $m \geq 1$ so that $\ker(\varphi^m) \cap \operatorname{im}(\varphi^m) = \{0\}$.
5. Suppose M/L and L/K are algebraic extensions of fields. Prove that M/K is algebraic. (You may not assume these extensions are finite.)
6. Let R be a ring with 1 and let M be a Noetherian right R-module. Let $f : M \to M$ be a surjective R-module homomorphism. Prove that f is also injective.
7. Let $A, B \in \mathbb{M}_n(\mathbb{Q})$ be two $n \times n$ matrices defined over $\mathbb{Q}$. If A and B are similar over $\mathbb{C}$, are they similar over $\mathbb{Q}$? Justify your answer.
8. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.
9. Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.
10. Find the Jordan Canonical Form for the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$