1. Let $G$ be a group of order $p^n$, where $p$ is a prime.
   (i) Show that the center of $G$ is non-trivial.
   (ii) Show that every maximal subgroup of $G$ is normal.

2. Let $G$ be a group of order $p^n m$, where $p$ is a prime and $\gcd(p, m) = 1$. Show that $G$ has a subgroup of order $p^n$.

3. Let $G$ be the group with presentation
   \[ G = \langle a, b | a^9, b^4, b^{-1}ab = a^5 \rangle. \]
   (i) Find the order of $G$.
   (ii) Find the center $Z$ of $G$.
   (iii) Find $G/Z$.

4. (i) Show that every a vector space has a basis. (Do not assume that $V$ is finite dimensional.)
   (ii) Let $V$ be a vector space of finite dimension and let $T : V \to V$ be a linear transformation. Show that $V = K \oplus W$ where $K = \ker(T)$ and $W \cong \text{Image}(T)$.

5. Show that if a finite ring $R$ with 1 admits an injective (ring) homomorphism from a field, then the number of elements of $R$ must be a power of a prime number. Is $R$ necessarily a field?

6. For a field $K$ and $n \geq 1$ let $J_{n,K}$ denote the $n \times n$ matrix over $K$ whose $(i,j)$ entry is equal to $(-1)^{i+j} \in K$.
   (a) Find the Jordan form of $J_{3,\mathbb{F}_2}$;
   (b) Find the Jordan form of $J_{3,\mathbb{F}_3}$;
   (c) Find the Jordan form of $J_{3,\mathbb{Q}}$.

7. Let $R$ be a PID. Show that every non-zero prime ideal is maximal.

8. Find the Galois group of the polynomial $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$.

9. Let $F$ be a field and let $f(x) \in F[x]$. Show that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over $F$.

10. An $R$-module $M$ is said to be **irreducible** if $M \neq \{0\}$ and $M$ has no $R$-submodules except $\{0\}$ and $M$. Let $V$ be a finite-dimensional vector space over a field $k$, and let $T : V \to V$ be a linear transformation. Then $T$ gives $V$ the structure of a $k[x]$-module. Prove that $V$ is irreducible as a $k[x]$-module if and only if the characteristic polynomial of $T$ is irreducible in $k[x]$.