Algebra PhD Qualifying Exam - Spring 2014

All rings have an identity element.

1. Let \( F_p \) be a field with \( p \) elements, \( p \) a prime. Let \( G = GL_2(F_p) \), the group invertible \( 2 \times 2 \) matrices with entries in \( F_p \).

   (a) Find a Sylow \( p \)-subgroup of \( G \).

   (b) Find (with proof) the number of Sylow \( p \)-subgroups of \( G \).

2. Let \( H \) be a normal subgroup of prime order \( p \) in a finite group \( G \). Suppose \( p \) is the smallest prime which divides the order of \( G \). Prove that \( H \) is contained in \( Z(G) \), the center of \( G \).

3. Let \( k \) be a field. Let \( I \) be the \( n \times n \) identity matrix with entries in \( k \), and let \( M \) be an \( n \times n \) matrix with entries in \( k \). Let \( x \) be an indeterminate. Prove that \( \det(I + xM) \equiv 1 + x \text{tr}(M) \pmod{x^2} \). Here \( \text{tr}(M) \) is the trace of \( M \). Recall that \( \text{tr}(M) \) is the sum of the diagonal entries.

4. Determine (with proof) all ideals in the formal power series ring \( k[[t]] \), \( k \) a field.

5. Prove that the following polynomials are irreducible in \( \mathbb{Q}[x] \).

   (a) \( x^5 + 6x + 12 \);

   (b) \( x^3 + 6x^2 + 7 \).

6. Let \( R \) be a nonzero commutative ring. Prove or disprove: If \( R^m \) is isomorphic to \( R^n \) as an \( R \)-module, then \( m = n \).

7. Give all possible complex \( 4 \times 4 \) matrices which can be Jordan canonical forms of \( 4 \times 4 \) matrices with real entries.

8. Find the 20th cyclotomic polynomial \( \phi_{20}(x) \in \mathbb{Z}[x] \).

9. Let \( F/\mathbb{Q} \) be a field extension, not necessarily of finite degree. Here \( \mathbb{Q} \) is the field of rational numbers. Suppose that for all \( \alpha \in F \), \( [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq n \). Prove that \( [F : \mathbb{Q}] \leq n \).

10. Let \( K \) be a splitting field over \( \mathbb{Q} \) of the polynomial \( x^5 - 2 \). Find all intermediate fields \( F \) between \( K \) and \( \mathbb{Q} \), and give generators over \( \mathbb{Q} \) for each.