Algebra PhD Qualifying Exam - Spring 2014

All rings have an identity element.

1. Let $\mathbb{F}_p$ be a field with p elements, p a prime. Let $G = GL_2(\mathbb{F}_p)$, the group invertible 2×2 matrices with entries in $\mathbb{F}_p$.
   (a) Find a Sylow p-subgroup of G.
   (b) Find (with proof) the number of Sylow p-subgroups of G.

2. Let H be a normal subgroup of prime order p in a finite group G. Suppose p is the smallest prime which divides the order of G. Prove that H is contained in $Z(G)$, the center of G.

3. Let k be a field. Let I be the $n \times n$ identity matrix with entries in k, and let M be an $n \times n$ matrix with entries in k. Let x be an indeterminate. Prove that $\det(I + xM) \equiv 1 + x\,\mathrm{tr}(M) \pmod{x^2}$. Here $\mathrm{tr}(M)$ is the trace of M. Recall that $\mathrm{tr}(M)$ is the sum of the diagonal entries.

4. Determine (with proof) all ideals in the formal power series ring $k[[t]]$, k a field.

5. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$.
   (a) $x^5 + 6x + 12$;
   (b) $x^3 + 6x^2 + 7$.

6. Let R be a nonzero commutative ring. Prove or disprove: If R^m is isomorphic to R^n as an R-module, then $m = n$.

7. Give all possible complex 4×4 matrices which can be Jordan canonical forms of 4×4 matrices with real entries

8. Find the 20th cyclotomic polynomial $\phi_{20}(x) \in \mathbb{Z}[x]$.

9. Let $\mathbb{F}/\mathbb{Q}$ be a field extension, not necessarily of finite degree. Here $\mathbb{Q}$ is the field of rational numbers. Suppose that for all $\alpha \in \mathbb{F}$, $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq n$. Prove that $[\mathbb{F} : \mathbb{Q}] \leq n$.

10. Let K be a splitting field over $\mathbb{Q}$ of the polynomial $x^5 - 2$. Find all intermediate fields F between K and $\mathbb{Q}$, and give generators over $\mathbb{Q}$ for each.