1. Let $G$ be a group of order $p^n$, where $p$ is prime. Show that, for each $0 \leq k \leq n$, the group $G$ has a normal subgroup of order $p^k$.

2. Construct a non-abelian group $G$ of order 21 and determine the sizes of the conjugacy classes of $G$.

3. Show that there is no simple group of order $3393 = 3^3 \cdot 13 \cdot 29$.

4. $x^4 + x^2 - 6$. Let $f(x) = x^3 + \frac{2}{7} \in \mathbb{Q}[x]$.
   (i) Find a polynomial $g(x) \in \mathbb{Z}[x]$ that has the same Galois group as $f(x)$.
   (ii) Find the Galois group of $g(x)$.

5. Let $\alpha = \sqrt{7} + 3\sqrt{5}$. Find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, and find $(1 + \alpha)^{-1}$ in the form $a + b\alpha + c\alpha^2 + \ldots$, where $a, b, c, \cdots \in \mathbb{Q}$.

6. Let $R$ be a commutative ring with 1. Find the center of $M_n(R)$? Justify your answer.

7. Let $N$ be a positive integer. Let $x$ be an integer relatively prime to $N$, $d$ relatively prime to $\varphi(N)$, and $dd' \equiv 1 \mod \varphi(N)$. Show that $y \equiv x^d \mod N$ implies that $x \equiv y^{d'} \mod N$.

8. Prove that $(x - 1)(x - 2) \cdots (x - n) - 1$ is irreducible over $\mathbb{Z}$ for all $n \geq 1$.

9. Let $R$ be a ring with 1, and let $M$ be a left $R$-module. The set of torsion elements is denoted $\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R \}$.
   (a) Prove that if $R$ is an integral domain then $\text{Tor}(M)$ is a submodule of $M$.
   (b) Give an example of a ring $R$ and an $R$-module $M$ such that $\text{Tor}(M)$ is not a submodule.

10. Let $V$ be a vector space of finite dimension over a field $F$. If $\varphi$ is any linear transformation from $V$ to $V$, prove there is an integer $m$ such that $\ker \varphi^m \cap \im \varphi^m = \{0\}$. 