## PH. D. QUALIFYING EXAM WINTER 2010 - ALGEBRA

Answer all the questions. Here $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the finite field with $p$ elements and $\mathbb{F}_{p^n}$ is the finite field with $p^n$ elements.

1. Show that any finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic.

2. Prove Cauchy's theorem: let $G$ be a finite group where $|G| = n$ and let $p$ be a prime dividing $n$. Then there is an element in $G$ of order $p$.

3. Show that a group of order $pqr$, where $p < q < r$ are primes, cannot be simple.

4. Let $M \in GL(n, \mathbb{C})$ be a matrix of finite order. Show that $M$ is diagonalizable. Is the same result true for matrices in $GL(n, F)$, where $F$ is any field?

5. Construct the field $F = \mathbb{F}_{125}$ with 125 elements. How many primitive elements are there in the extension $F/\mathbb{F}_5$? Here, a primitive element is $\alpha \in F$ such that $\mathbb{F}_5(\alpha) = F$.

6. Let $V = \mathbb{C}[x]$, considered as a vector space over $\mathbb{C}$. Let $D = \frac{d^2}{dx^2} - \frac{d}{dx}$ be a linear transformation of $V$. Let $W$ be the smallest subspace of $V$ that contains the element $x^3$ and which is invariant under $D$. Find $W$ and then calculate the Jordan Canonical form of $D|_W$ and of $D|_W + I$, where $I$ is the identity function on $W$ and $D|_W$ is the restriction of $D$ to $W$.

7. Let $A$ be a finite abelian group of order $n = p^k m$, where $p$ is a prime and $\gcd(p, m) = 1$. Show that

$$(\mathbb{Z}/p^k\mathbb{Z}) \otimes_{\mathbb{Z}} A$$

is isomorphic to the Sylow $p$-subgroup of $A$.

8. Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

9. Prove that in a unique factorization domain a non-zero element is a prime if and only if it is irreducible.

10. Find the Galois group of:
(1) $x^4 + 1 \in \mathbb{Q}[x]$;
(2) $x^4 + 1 \in \mathbb{R}[x]$;
(3) the extension $\mathbb{F}_{16}/\mathbb{F}_2$.