Answer all questions. Partial credit will be given

(1) Let \( G \) be a finite group, \( H \) a subgroup of \( G \), and \( N \) a normal subgroup of \( G \). Show that if the order of \( H \) is relatively prime to the index of \( N \) in \( G \), then \( H \subseteq N \).

(2) Let \( G \) be an abelian group. Let \( K = \{ a \in G : a^2 = 1 \} \) and let \( H = \{ x^2 : x \in G \} \). Show that \( G/K \cong H \).

(3) Prove that there is no simple group of order 80.

(4) Give an example (with proof) of a Galois extension \( K \) of \( \mathbb{Q} \) such that \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/14\mathbb{Z} \).

(5) Let \( p \) be a prime that is congruent to 2 mod 5, and let \( F = \mathbb{F}_p \) be the finite field with \( p \) elements. Prove that the polynomial
\[
f(x) = x^4 + x^3 + x^2 + x + 1 \in F[x]
\]
is irreducible over \( F \).

(6) Denote by \( M_3(\mathbb{Q}) \) the set of \( 3 \times 3 \) matrices with entries in \( \mathbb{Q} \). Let \( \varphi : M_3(\mathbb{Q}) \to M_3(\mathbb{Q}) \) be the map sending \( m \in M_3(\mathbb{Q}) \) to \( \varphi(m) = m^2 + 2m + 2 \). Show that \( \varphi(m) \neq 0 \) for all \( m \in M_3(\mathbb{Q}) \).

(7) Let \( k \) be a field, \( R \) a nonzero ring, and \( \varphi : k \to R \) a nonzero ring map. Show that \( \varphi \) is injective.

(8) Let \( R := R' \times R'' \) be a product of commutative rings with 1. Prove that \( R \) is a domain if and only if one of \( R' \) and \( R'' \) is the zero ring, and the other is a domain.

(9) Let \( R := R' \times R'' \) be a product of commutative rings with 1, and let \( I \subset R \) be an ideal. Show that \( I = I' \times I'' \) for some ideals \( I' \in R' \) and \( I'' \in R'' \), and that \( R/I = R'/I' \times R''/I'' \).

(10) Let \( R \) be a PID, and let \( x, y \in R \). Recall that for elements \( a, b \in R \), \( a \) is a factor of \( b \) if and only if there is some \( c \in R \) with \( b = ac \). Denote by \( \langle x \rangle \) and \( \langle y \rangle \) the ideals of \( R \) generated by \( x \) and \( y \), respectively. Prove that \( x \) and \( y \) share no nonunit factors if and only if \( \langle x \rangle \) and \( \langle y \rangle \) are comaximal in \( R \).