Algebra qualifying exam
January 2015

Answer all questions. Partial credit will be given

(1) Let G be a finite group, H a subgroup of G, and N a normal subgroup of G. Show that if the order of H is relatively prime to the index of N in G, then $H \subseteq N$.

(2) Let G be an abelian group. Let $K = \{a \in G : a^2 = 1\}$ and let $H = \{x^2 : x \in G\}$. Show that $G/K \cong H$.

(3) Prove that there is no simple group of order 80.

(4) Give an example (with proof) of a Galois extension K of $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/14\mathbb{Z}$.

(5) Let p be a prime that is congruent to 2 mod 5, and let $F = \mathbb{F}_p$ be the finite field with p elements. Prove that the polynomial

$$f(x) = x^4 + x^3 + x^2 + x + 1 \in F[x]$$

is irreducible over F.

(6) Denote by $M_3(\mathbb{Q})$ the set of 3×3 matrices with entries in $\mathbb{Q}$. Let $\varphi : M_3(\mathbb{Q}) \to M_3(\mathbb{Q})$ be the map sending $m \in M_3(\mathbb{Q})$ to $\varphi(m) = m^2 + 2m + 2$. Show that $\varphi(m) \neq 0$ for all $m \in M_3(\mathbb{Q})$.

(7) Let k be a field, R a nonzero ring, and $\varphi : k \to R$ a nonzero ring map. Show that φ is injective.

(8) Let $R := R' \times R''$ be a product of commutative rings with 1. Prove that R is a domain if and only if one of R' and R'' is the zero ring, and the other is a domain.

(9) Let $R := R' \times R''$ be a product of commutative rings with 1, and let $I \subset R$ be an ideal. Show that $I = I' \times I''$ for some ideals $I' \in R'$ and $I'' \in R''$, and that $R/I = R'/I' \times R''/I''$.

(10) Let R be a PID, and let $x, y \in R$. Recall that for elements $a, b \in R$, a is a factor of b if and only if there is some $c \in R$ with $b = ac$. Denote by $\langle x \rangle$ and $\langle y \rangle$ the ideals of R generated by x and y, respectively. Prove that x and y share no nonunit factors if and only if $\langle x \rangle$ and $\langle y \rangle$ are comaximal in R.