Algebra Ph.D. Qualifying Exam

May 2015

Answer all 10 questions. Your judgment as to which theorems are appropriate to use for each problem is part of what is being tested, so be sure not to use theorems which make the problems trivial.

(1) Let G be a finite group, and let H and K be subgroups. Define the set

$$HK = \{hk : h \in H, k \in K\}.$$

Prove that

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

(2) Prove that no group of order 72 is simple.

(3) Determine (with proof) all finite groups G such that the automorphism group of G is the trivial group.

(4) Let R be a commutative ring with $1 \neq 0$. Prove that an ideal I of R is maximal if and only if R/I is a field.

(5) Let $R = \mathbb{Z}/3\mathbb{Z}$. Find (with justification) all $a \in R$ such that the quotient ring

$$R[x]/(x^3 + x^2 + ax + 1)$$

is a field.

(6) Let R be a commutative ring with 1. Prove the equivalence of the following two conditions:
   (a) Every ideal of R is finitely generated.
   (b) If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an ascending chain of ideals of R, then there is an integer N such that $I_k = I_N$ for all $k \geq N$.

(7) Let $G = \mathbb{Z} \oplus \mathbb{Z}$ be a free abelian group of rank 2, and let H be the subgroup of G generated by the two elements $(3, 0)$ and $(4, 2)$ in G. Determine the structure of G/H as a product of cyclic groups (i.e. write $G/H \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$ for some integers $n_1, \ldots, n_k$).

(8) Let L/K be an extension of fields of finite degree, with $[L : K]$ relatively prime to 6. Let $u \in L$. Prove that $K(u) = K(u^3)$.

(9) Let F be a finite field with 121 elements, and let $f(x) = x^{12} + x^8 + x^4 + 1 \in F[x]$. How many roots of $f(x)$ are contained in F? Be sure to justify (prove) your answer.

(10) Let $\zeta = e^{2\pi i/12}$ be a primitive twelfth root of unity, and let $K = \mathbb{Q}(\zeta)$. Find all subfields of $K/\mathbb{Q}$, and write each of them as $\mathbb{Q}(\alpha)$, for some $\alpha \in K$.