Algebra Ph.D. Qualifying Exam

May 2017

Answer all 10 questions. Your judgment as to which theorems are appropriate to use for each problem is part of what is being tested, so be sure not to use theorems which make the problems trivial.

1. Prove that the alternating group A_5 is a simple group.

2. (a) Prove that a group of order 143 must be cyclic.
   (b) Determine all abelian groups of order 144.

3. Let V be a vector space. A linear operator $P : V \to V$ is called a *projection* if $P^2 = P$. Prove that if P is a projection, then the eigenvalues of P are either 0 or 1. (Do not assume that V is finite dimensional.)

4. Show that a 2×2 matrix over $\mathbb{C}$ satisfies its characteristic equation. (Do not use the Cayley-Hamilton Theorem.)

5. Let J be the 3×3 matrix
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$
   (a) Find the Jordan canonical form of J considered as a matrix with entries in $\mathbb{F}_2$.
   (b) Find the Jordan canonical form of J considered as a matrix with entries in $\mathbb{F}_3$.
   (c) Find the Jordan canonical form of J considered as a matrix with entries in $\mathbb{Q}$.

6. Prove Cauchy's theorem: Let G be a finite group where $|G| = n$ and let p be a prime dividing n. Then there is an element in G of order p.

7. Let R be a commutative ring with $1 \neq 0$. Prove that if every proper ideal of R is prime, then R is a field.

8. Let L/F be a field extension, and let $\alpha, \beta \in L$ be algebraic over f. Let f be the minimal polynomial of α over F, and let g be the minimal polynomial of β over F. Prove that f is irreducible over $F(\beta)$ if and only if g is irreducible over $F(\alpha)$.

9. An idempotent element of a ring R is an element $a \in R$ such that $a^2 = a$. Suppose that R is a commutative ring with $1 \neq 0$, and R contains an idempotent element $a \neq 0, 1$. Prove that every prime ideal in R contains an idempotent element not equal to 0 or 1.

10. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5 with Galois group S_5. Let $L/\mathbb{Q}$ be a splitting field of f, and let S be the set of all intermediate fields K between $\mathbb{Q}$ and L with $[K : \mathbb{Q}] = 24$.
    (a) Determine $|S|$.
    (b) Let F be the intersection of all the fields in S. Prove that $F \neq \mathbb{Q}$.