1. Let \( 0 < y_1 < x_1 \) and set
\[
x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n}, \quad n = 1, 2, \ldots
\]
(a) Prove that \( 0 < y_n < x_n \) \( (n = 1, 2, \ldots) \)
(b) Prove that \( y_n \) is increasing and bounded above, and \( x_n \) is decreasing and bounded below.
(c) Prove that \( 0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/2^n \) for \( n = 1, 2, \ldots \)
(d) Prove that \( \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \).

2. Prove that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

3. Let \( X \) be a complete metric space and let \( Y \) be a subspace of \( X \). Prove that \( Y \) is complete if and only if it is closed in \( X \).

4. Consider the subset
\[
H = \{(a, b, c, d, e) \in \mathbb{R}^5 \mid ax^4 + bx^3 + cx^2 + dx + e = 0 \text{ for some } x \in \mathbb{R}\}.
\]
(a) Prove that \( (1, 2, -4, 3, -2) \) is an interior point of \( H \).
(b) Find a point in \( H \) that is not an interior point. Justify your answer.

5. Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable at \( a \) and \( f(a) \neq 0 \). Let \( T = -Df(a)/f^2(a) \). Show that the equation
\[
\frac{1}{f(a + h)} - \frac{1}{f(a)} - T(h) = \frac{f(a) - f(a + h) + Df(a)(h)}{f(a)f(a + h)} + \frac{(f(a + h) - f(a))Df(a)(h)}{f^2(a)f(a + h)}
\]
makes sense, and is correct, for small \( h \). Deduce that \( D \left( \frac{1}{f} \right)(a) = T \).

6. Find the second order Taylor polynomial \( P(x, y, z) \) for the function \( f(x, y, z) = e^{2x+y-z} \) at the point \( (1, -1, 1) \).

7. If \( f : [a, b] \to \mathbb{R} \) is decreasing, prove that \( f \) is (Riemann-Darboux-) integrable on \( [a, b] \).

8. If \( T = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\} \), compute the integral on \( T \) of the function \( (x, y) \mapsto e^{(x-y)/(x+y)} \). (Hint: Change variables.)

9. Evaluate \( \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} \).

10. Prove that the equation \( z = 2 - e^{-z} \) has exactly one solution in the right half plane. Why must this solution be real?