MASTER'S EXAM, ANALYSIS, FEBRUARY 2011

1. Let $0 < y_1 < x_1$ and set

$$x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n}, \; n = 1, 2, \ldots$$

   (a) Prove that $0 < y_n < x_n \quad (n = 1, 2, \ldots)$

   (b) Prove that y_n is increasing and bounded above, and x_n is decreasing and bounded below.

   (c) Prove that $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/2^n$ for $n = 1, 2, \ldots$

   (d) Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

2. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

3. Let X be a complete metric space and let Y be a subspace of X. Prove that Y is complete if and only if it is closed in X.

4. Consider the subset

$$H = \{(a, b, c, d, e) \in \mathbb{R}^5 \mid ax^4 + bx^3 + cx^2 + dx + e = 0 \text{ for some } x \in \mathbb{R}\}.$$

   (a) Prove that $(1, 2, -4, 3, -2)$ is an interior point of H.

   (b) Find a point in H that is not an interior point. Justify your answer.

5. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\boldsymbol{a}$ and $f(\boldsymbol{a}) \neq 0$. Let $T = -Df(\boldsymbol{a})/f^2(\boldsymbol{a})$. Show that the equation

$$\frac{1}{f(\boldsymbol{a}+\boldsymbol{h})} - \frac{1}{f(\boldsymbol{a})} - T(\boldsymbol{h}) = \frac{f(\boldsymbol{a}) - f(\boldsymbol{a}+\boldsymbol{h}) + Df(\boldsymbol{a})(\boldsymbol{h})}{f(\boldsymbol{a})f(\boldsymbol{a}+\boldsymbol{h})} + \frac{(f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}))Df(\boldsymbol{a})(\boldsymbol{h})}{f^2(\boldsymbol{a})f(\boldsymbol{a}+\boldsymbol{h})}$$

   makes sense, and is correct, for small $\boldsymbol{h}$. Deduce that $D\left(\frac{1}{f}\right)(\boldsymbol{a}) = T$.

6. Find the second order Taylor polynomial $P(x, y, z)$ for the function $f(x, y, z) = e^{2x+y-z}$ at the point $(1, -1, 1)$.

7. If $f : [a, b] \to \mathbb{R}$ is decreasing, prove that f is (Riemann-Darboux-) integrable on $[a, b]$.

8. If $T = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$, compute the integral on T of the function $(x, y) \mapsto e^{(x-y)/(x+y)}$. (Hint: Change variables.)

9. Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta}$.

10. Prove that the equation $z = 2 - e^{-z}$ has exactly one solution in the right half plane. Why must this solution be real?