MASTER'S EXAM, ANALYSIS, JANUARY 2011

1. When we say that $(\mathbb{R}, +, \cdot, \leq)$ is a complete ordered field, we are asserting more than that $(\mathbb{R}, +, \cdot)$ is a field and $(\mathbb{R}, \leq)$ is a totally ordered set. What else are we asserting? (Be specific.)

2. Prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to A and $\{b_n\}_{n=1}^{\infty}$ is a sequence of nonzero real numbers converging to $B \neq 0$, then $\{a_n/b_n\}_{n=1}^{\infty}$ converges to A/B.

3. Let $f : X \to Y$ be a continuous function between metric spaces. If X is compact, prove that f is uniformly continuous.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function satisfying $\|f(\boldsymbol{x})\| < \|\boldsymbol{x}\|$ for every $\boldsymbol{x} \neq \mathbf{0}$. For some initial point $\boldsymbol{x}_0 \in \mathbb{R}^n$ define the sequence $\{\boldsymbol{x}_k\}_{k=0}^{\infty}$ recursively by the rule $\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k)$. Show that the sequence converges to $\mathbf{0}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$. Prove that there exists positive constants a and b such that
$$|f(x)| \leq a|x| + b, \quad \forall x \in \mathbb{R}.$$

6. Let $U \subset \mathbb{R}^2$ be a nonempty open set. Prove that there is no one-to-one continuously-differentiable function mapping U into $\mathbb{R}$.

7. Let $f : [0, 1] \to \mathbb{R}$ be defined by the formula
$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms.} \end{cases}$$
Determine with proof the value of the (Riemann-Darboux) integral $\int_0^1 f(x)\,dx$ or that the integral is undefined.

8. Evaluate $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \int_{x^3}^1 \sqrt{x^3 + z}\,dz\,dx\,dy.$

9. For $a > b > 0$, evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2 + a^2)(x^2 + b^2)}.$

10. Suppose that f is analytic in $|z| < 1$ and that $|f(z)| < \dfrac{1}{1 - |z|}$ for $|z| < 1$. Show that for $0 < R < 1$,
$$|f^{(n)}(0)| \leq \frac{1}{R^n(1 - R)} \quad (n = 1, 2, \ldots).$$
What choice of R yields the best upper bound?