Ph.D. QUALIFIER EXAMINATION: ANALYSIS

Fall 2013

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\mathcal{B}_X$ – the Borel σ-algebra in X
4. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
5. a.e.$[\mu]$ – almost every with respect to the measure μ
6. m – Lebesgue measure on $\mathbb{R}^k$
7. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
8. $\|f\|_\infty$ – the essential supremum of f
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dm(y)$
17. $C_c(X)$ – the continuous complex functions on X with compact support
18. $C_0(X)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dm(x)$ – the Fourier transform

Questions

1. State and prove Lebesgue's Dominated Convergence Theorem [You may assume Fatou's Lemma in your proof.]

2. State Lusin's Theorem, and use it to prove the following: For a measure space $(X, \mathcal{M}, \mu)$ where X is a locally compact Hausdorff space, $\mathcal{M}$ contains $\mathcal{B}_X$, and μ is a complete positive measure that is outer regular on all measurable sets, inner regular on all open sets and all measurable sets of finite measure, if f is a complex measurable function on X with $|f| \leq 1$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$ such that $f(x) = 0$ for all $x \notin A$, then there is a sequence $g_n \in C_c(X)$ such that $|g_n| \leq 1$ and $g_n(x) \to f(x)$ as $n \to \infty$ a.e.$[\mu]$.

3. Prove that if X is a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$ with respect to metric defined by the supremum norm $\|f\| = \sup_{x \in X} |f(x)|$. [You may assume Urysohn's Lemma in your proof.]

4. Prove that if L is a continuous linear functional on a Hilbert space H with inner product $(\cdot, \cdot)$, then there exists a unique $y \in H$ such that $Lx = (x, y)$ for all $x \in H$.

5. State the Opening Mapping Theorem for Banach spaces, and use it to prove the following: If X and Y are Banach spaces and if $\Lambda : X \to Y$ is a bijective bounded linear transformation, then there exists $\delta > 0$ such that

$$\|\Lambda x\| \geq \delta \|x\|.$$

6. State the Theorem of Lebesgue-Radon-Nikodym, and give an example where one of the conclusions of the Theorem of Lebesgue-Radon-Nikodym fails to hold.

7. Prove there is a continuous real-valued function of a real variable that is differentiable almost everywhere, for which the Fundamental Theorem of Calculus fails to hold.

8. State Fubini's Theorem. Prove that there is a real-valued function $f(x, y)$ defined on $[0, 1] \times [0, 1]$, with Lebesgue measure on each $[0, 1]$, for which both iterated integrals of f exist but are not equal.

9. Prove that if $h = f * g$ for $f, g \in L^1(\mathbb{R})$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$. [You may assume that $f(x - y)g(y)$ is $L^1(\mathbb{R}^2)$ and Fubini's Theorem in your proof.]

10. For $f \in L^1(\mu)$, prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\int_E |f| \, d\mu < \epsilon$$

whenever $\mu(E) < \delta$.