Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Fall 2015

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.
Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\mathcal{B}_X$ – the Borel σ-algebra in X
4. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
5. a.e.$[\mu]$ – almost every with respect to the measure μ
6. m – Lebesgue measure on $\mathbb{R}^k$
7. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
8. $\|f\|_\infty$ – the essential supremum of f
9. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
10. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
12. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
13. $|\lambda|$ – the total variation of a measure λ.
14. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
15. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
16. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
17. Lip α – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
18. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dm(y)$
19. $C_c(X)$ – the continuous complex functions on X with compact support
20. $C_0(X)$ – the continuous complex functions on a LCH space X which vanish at infinity
21. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform

Questions

1. State and prove Lebesgue's Dominated Convergence Theorem. [You may assume Fatou's Lemma in your proof.]

2. State and prove Lusin's Theorem.

3. Suppose f is a complex measurable function on X, μ a positive measure on X, and $\varphi(p) = \int_X |f|^p d\mu = \|f\|_p^p$. Let $E = \{p : 0 < p < \infty, \varphi(p) < \infty\}$. Assume that $\|f\|_\infty > 0$. (a) Prove that $\log \varphi$ is convex in the interior of E. (b) If $r < p < s$, prove that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$ and that $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

4. Let μ be a positive measure on X with $\mu(X) < \infty$. If $f_n \to f$ in measure, prove that $\{f_n\}$ has a subsequence which converges to f a.e. $[\mu]$. [A sequence of complex measurable function functions $\{f_n\}$ on X is said to converge in measure to a measurable function f if for every $\epsilon > 0$ there corresponds a positive integer N such that $\mu\{x : |f_n(x) - f(x)| > \epsilon\} < \epsilon$ for all $n > N$.]

5. Suppose μ is a positive measure on X with $\mu(X) < \infty$. For $f_n \in L^1(\mu)$, $n = 1, 2, 3, \ldots$, suppose there exists $p > 1$ and $C < \infty$ such that

$$\int_X |f_n|^p d\mu < C \text{ for all } n = 1, 2, 3, \ldots.$$

Prove for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $n = 1, 2, 3, \ldots$, there holds $|\int_E f_n d\mu| < \epsilon$ whenever $\mu(E) < \delta$.

6. Suppose p and q are conjugate exponents with $1 < p < \infty$, and set $\alpha = 1/q$. Prove that if f is absolutely continuous on $[a, b]$ and $f' \in L^p$, then $f \in \text{Lip } \alpha$.

7. If $f \in L^1(\mathbb{R})$, prove that $\limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)| dy = 0$ for almost every $x \in \mathbb{R}$. [You may assume the denseness of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$ in your proof.]

8. Let X and Y be topological spaces and $X \times Y$ the Cartesian product endowed with the product topology. (a) Suppose $f : X \to Y$ is continuous. Prove that $E \in \mathcal{B}_Y$ implies $f^{-1}(E) \in \mathcal{B}_X$. (b) Suppose $A \in \mathcal{B}_X$ and $E \in \mathcal{B}_Y$. Prove that $A \times E \in \mathcal{B}_{X \times Y}$.

9. Find the value of

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx$$

and justify your answer.

10. Compute $\int_0^\infty \cos(x^2) dx$.