MASTER'S EXAM, ANALYSIS, AUGUST 2010

1. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers converging to 0, prove that the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges, or give a specific counterexample.

2. Prove directly (without using any theorem about absolute convergence) that every rearrangement of the series $1 + 1/2 + 1/4 + 1/8 + \cdots$ converges to 2.

3. Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

4. Prove the Intermediate Value Theorem: If f is a real-valued continuous function on the interval $[a, b]$ with $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

5. State and prove the Mean-Value Theorem for functions of a single variable.

6. Let $B(r, \mathbf{0}) \subset \mathbb{R}^n$ denote the open ball of radius $r > 0$, centered at $\mathbf{0}$. Let $f : B(r, \mathbf{0}) \to \mathbb{R}$ and suppose there exists $\alpha > 1$ such that

$$|f(\boldsymbol{x})| \leq \|\boldsymbol{x}\|^{\alpha} \quad \forall \boldsymbol{x} \in B(r, \mathbf{0}).$$

   Prove that f is differentiable at $\mathbf{0}$. What happens to this result when $\alpha = 1$?

7. The colatitude of a point (x, y, z) on the hemisphere

$$H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$$

   is the angle between the vector (x, y, z) and the vector $(0, 0, 1)$. What is the average colatitude of a point on H (with respect to surface area)?

8. Let $H = [a, b] \times [c, d]$ and suppose that $f : H \to \mathbb{R}$ is continuous and that $g : [a, b] \to \mathbb{R}$ is Riemann integrable. Prove that

$$F(y) = \int_a^b g(x) f(x, y) dx$$

   is continuous on $[c, d]$.

9. Prove that for any polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ we have $\sup_{|z|=1} |P(z)| \geq 1$. *Hint*: Consider $Q(0)$, where $Q(z) = z^n P(z^{-1})$.

10. Prove: If $f(z)$ and $|f(z)|$ are both analytic on a connected domain D, then f is constant in D.