Qualifying Exam In Analysis August 2010
Pick three of the following. Identify which ones you are doing.

1. Let $f \in L^1(\Omega, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a probability space $(P(\Omega) = 1)$. Also let $\mathcal{G}$ be a σ algebra of sets, $\mathcal{G} \subseteq \mathcal{F}$. Show there exists a unique function, denoted by $E(f|\mathcal{G})$ which is $\mathcal{G}$ measurable, and for all $A \in \mathcal{G}$,
$$\int_A f d\mu = \int_A E(f|\mathcal{G}) \, d\mu$$

2. In the situation of Problem 1, show that if f, g are functions in $L^1(\Omega, \mu)$ and $f \leq g$, then $E(f|\mathcal{G}) \leq E(g|\mathcal{G})$ a.e. and that the map $f \to E(f|\mathcal{G})$ is linear.

3. Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces, but there exist examples where the Vitali convergence theorem applies but the dominated convergence theorem cannot be applied.

4. A sequence of functions $\{f_n\}$ on a measure space $(\Omega, \mathcal{F}, \mu)$ is said to converge to 0 in measure if
$$\lim_{n\to\infty} \mu([x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon]) = 0$$
for each fixed $\varepsilon > 0$. Show that there exists a subsequence $\{f_{n_k}\}$ such that f_{n_k} converges pointwise to 0 off a set of measure zero.

5. A random variable is a measurable function $X : (\Omega, \mathcal{F}, P) \to \mathbb{R}^n$ where $P(\Omega) = 1$. Show there exists a unique Radon measure called the distribution measure, λ_X defined on a σ algebra of sets of $\mathbb{R}^n$ containing the Borel sets such that whenever E is a Borel set in $\mathbb{R}^n$,
$$\lambda_X(E) = P([X \in E]).$$
Part of this is to show that $[X \in E]$ is in $\mathcal{F}$ whenever E is Borel.

6. Let $\{f_n\}$ be a sequence of functions defined on a finite measure space (Ω, μ) meaning $(P(\Omega) < \infty)$. Also suppose there exists a constant C such that for some $p > 1$, $||f_n||_{L^p} < C$. Show this collection of functions is uniformly integrable.

Pick 7 of the following Identify which ones you are doing

1. Let $f \in L^p(\mathbb{R}^n), p > 1$, with respect to standard Lebesgue measure. Show that
$$\lim_{\mathbf{y}\to\mathbf{0}} \left(\int_{\mathbb{R}^n} |f(\mathbf{x}) - f(\mathbf{x} - \mathbf{y})|^p \, dx \right)^{1/p} = 0$$
Your proof should be based on standard facts about Lebesgue measure and advanced calculus.

2. Let $\{\phi_n\}$ be a mollifier. Recall that this means $\int \phi_n = 1$, ϕ_n is infinitely differentiable, and the support of ϕ_n is contained in $B(\mathbf{0}, a_n)$ where $\lim_{n\to\infty} a_n = 0$. Show that if $f \in L^p(\mathbb{R}^n)$, then
$$\lim_{n\to\infty} ||f - f * \phi_n||_{L^p(\mathbb{R}^n)} = 0$$
(The measure is ordinary Lebesgue measure). Also show that $f * \phi_n$ is infinitely differentiable. Your argument should be based on standard theorems about the Lebesgue integral and Lebesgue measure.

3. Suppose $|f(t)| \leq Ce^{rt}$ for some r. and that $f \in C_0([0, \infty))$, the space of continuous functions which vanishes outside a compact set. Show that if
$$\int_0^\infty e^{-st} f(t) \, dt = 0$$
for each s sufficiently large, then $f(t) = 0$ for all t.

4. Prove the Riemann Lebesgue lemma which says that if $f \in L^1(\mathbb{R})$, then

$$\lim_{r \to \infty} \int_{\mathbb{R}} \sin(ru) f(u)\, du = 0$$

5. Let E be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if

$$\lim_{r \to 0} \frac{m_n(E \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} = 1.$$

Show that a.e. point of E is a point of density. **Hint:** The numerator of the above quotient is $\int_{B(\mathbf{x},r)} \mathcal{X}_E(\mathbf{x})\, dm$. Now consider the fundamental theorem of calculus.

6. Let f be in $L^1_{loc}(\mathbb{R}^n)$. Show Mf is Borel measurable. **Hint:** First consider the function,

$$F_r(\mathbf{x}) \equiv \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} |f(\mathbf{x})|\, dm_n$$

Argue F_r is continuous.

7. Give an example of a function defined on an interval of $\mathbb{R}$ which is strictly increasing and has the property that its derivative equals 0 on a set of positive measure.

8. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ with respect to ordinary Lebesgue measure for $p \geq 1$, show that $f * g(x)$ exists for a.e. x. Also show that $||f * g||_{L^p} \leq ||f||_{L^1} ||g||_{L^p}$. If you need to use a theorem, be sure to explain why the theorem applies.

9. If $f \in L^p, 1 < p < \infty$, show $Mf \in L^p$. (You can use that Mf is Borel measurable). Show the following estimate.

$$||Mf||_p \leq A(p, n)||f||_p.$$

Hint: Let

$$f_1(\mathbf{x}) \equiv \begin{cases} f(\mathbf{x}) \text{ if } |f(\mathbf{x})| > \alpha/2, \\ 0 \text{ if } |f(\mathbf{x})| \leq \alpha/2. \end{cases}$$

Argue $[Mf(\mathbf{x}) > \alpha] \subseteq [Mf_1(\mathbf{x}) > \alpha/2]$. Then use the distribution function. Recall why

$$\int (Mf)^p dx = \int_0^\infty p\alpha^{p-1} m([Mf > \alpha]) d\alpha$$

$$\leq \int_0^\infty p\alpha^{p-1} m([Mf_1 > \alpha/2]) d\alpha.$$

Now use the fundamental estimate satisfied by the maximal function and Fubini's Theorem as needed.

10. If E has positive Lebesgue measure, show that

$$E - E \equiv \{x - y : x \in E \text{ and } y \in E\}$$

must contain an interval of the form $(-\delta, \delta)$. State clearly all theorems used to show this.