Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Fall 2011

Instructions: Answer exactly 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. \( \mathbb{R}^k \) – Euclidean \( k \)-dimensional space
2. \( \mathbb{C} \) – the complex numbers
3. \((X, \mathcal{M}, \mu)\) – a measure space where \( X \) is a set, \( \mathcal{M} \) is a \( \sigma \)-algebra of subsets of \( X \), and \( \mu \) is a measure on \( \mathcal{M} \)
4. \( a.e. [\mu] \) – almost every with respect to the measure \( \mu \)
5. \( m \) – Lebesgue measure on \( \mathbb{R}^k \)
6. \( \|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p} \) – the \( L^p \)-norm of a \( \mu \)-measurable function \( f : X \to \mathbb{C} \)
7. \( \|f\|_\infty \) – the essential supremum of \( f \)
8. \( p, q \) – conjugate exponents where \( \frac{1}{p} + \frac{1}{q} = 1 \)
9. \( L^p(\mu) \) – the space of \( \mu \)-measurable functions \( f : X \to \mathbb{C} \) with \( \|f\|_p < \infty \)
10. \( L^p(\mathbb{R}^k) \) – the space of Lebesgue measurable functions \( f : \mathbb{R}^k \to \mathbb{C} \) with \( \|f\|_p < \infty \)
11. \( \|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\} \) – operator norm of a linear transformation \( \Gamma : X \to Y \) where \( X \) and \( Y \) are normed linear spaces
12. \( |\lambda| \) – the total variation of a measure \( \lambda \)
13. \( \lambda \ll \mu \) – the measure \( \lambda \) is absolutely continuous with respect to the measure \( \mu \)
14. \( \lambda \perp \mu \) – the measures \( \lambda \) and \( \mu \) are mutually singular
15. \( \frac{d\lambda}{d\mu} \) – the Radon-Nikodym derivative of \( \lambda \) with respect to \( \mu \) where \( \lambda \ll \mu \)
16. \( \text{Lip } \alpha \) – the space of complex functions \( f \) on \([a, b]\) for which \( \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \); here \( 0 < \alpha \leq 1 \)
17. \( f \ast g \) – the convolution of \( f \) and \( g \): \( (f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy \)
18. \( C_0(\mathbb{R}^k) \) – the continuous complex functions on \( \mathbb{R}^k \) which vanish at infinity
19. \( \hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dx \) – the Fourier transform
Questions

1. State and prove Lebesgue’s Dominated Convergence Theorem. [You may assume Fatou’s Lemma in your proof.]

2. Let \( X \) be a locally compact Hausdorff space in which every open set is \( \sigma \)-compact. If \( \lambda \) is a positive Borel measure on \( X \) such that \( \lambda(K) < \infty \) for every compact subset \( K \) of \( X \), then \( \lambda \) is regular. [Hint: there is a regular positive Borel measure \( \mu \) on \( X \) such that \( \int_X f \, d\lambda = \int_X f \, d\mu \) for all continuous \( f \) on \( X \) with compact support; show that \( \lambda = \mu \). You may assume Urysohn’s Lemma and the Monotone Convergence Theorem in your proof.]

3. For a positive measure \( \mu \) prove that if \( r < p < s \), then \( \| f \|_p \leq \max\{\| f \|_r, \| f \|_s \} \) for every complex measurable function \( f \). [Hint: for \( \phi(p) = \| f \|_p \), the function \( \log \phi(p) \) is convex in the interior of \( \{ p : \phi(p) < \infty \} \).]

4. Prove that if \( A \subset [0, 2\pi] \) is Lebesgue measurable, then
\[
\lim_{n \to \infty} \int_A \cos nx \, dx = \lim_{n \to \infty} \int_A \sin nx \, dx = 0.
\]

5. Let \( X \) be a normed linear space, and \( X^* \) its dual space equipped with the norm \( \| f \| = \sup\{|f(x)| : \| x \| \leq 1\} \). Prove that \( X^* \) is a Banach space.

6. Let \( L^\infty = L^\infty(m) \) where \( m \) is Lebesgue measure on \( I = [0, 1] \). Prove that there is a bounded linear functional \( \Lambda \neq 0 \) on \( L^\infty \) that is 0 on \( C(I) \) (the space of continuous functions defined on \( I \)).

7. Prove that if \( f \in \text{Lip} 1 \) on \([a, b]\), then \( f \) is absolutely continuous on \([a, b]\) and \( f' \in L^\infty \).

8. For \( f \in L^1(\mathbb{R}) \) and \( g \in L^p(\mathbb{R}) \) with \( 1 < p < \infty \), prove that \( f \ast g \) exists a.e., that \( f \ast g \in L^p(\mathbb{R}) \), and that \( \| f \ast g \|_p \leq \| f \|_1 \| g \|_p \). [You may assume Fubini’s Theorem in your proof.]

9. For a positive integer \( n \), find (with proof) the Fourier transform of \( \chi_{[-n,n]} \ast \chi_{[-1,1]} \), where \( \chi_{[a,b]} \) is the characteristic function of \([a,b]\).

10. Suppose \( f \) is an entire function, and that in every power series
\[
f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n,
\]
at least one coefficient is 0. Prove that \( f \) is a polynomial.