Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2012

Instructions: Answer exactly 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean $k$-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where $X$ is a set, $\mathcal{M}$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure $\mu$
5. $m$ – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$ – the $L^p$-norm of a $\mu$-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_{\infty}$ – the essential supremum of $f$
8. $p, q$ – conjugate exponents where $\frac{1}{p} + \frac{1}{q} = 1$
9. $L^p(\mu)$ – the space of $\mu$-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where $X$ and $Y$ are normed linear spaces
12. $|\lambda|$ – the total variation of a measure $\lambda$.
13. $\lambda \ll \mu$ – the measure $\lambda$ is absolutely continuous with respect to the measure $\mu$
14. $\lambda \perp \mu$ – the measures $\lambda$ and $\mu$ are mutually singular
15. $\frac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ where $\lambda \ll \mu$
16. $\text{Lip}_\alpha$ – the space of complex functions $f$ on $[a, b]$ for which $\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
17. $f \ast g$ – the convolution of $f$ and $g$: $(f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dx$ – the Fourier transform
Questions


2. Construct a sequence of continuous functions $f_n$ on $[0,1]$ such that $0 \leq f_n \leq 1$, such that
   \[ \lim_{n \to \infty} \int_0^1 f_n(x) dx = 0, \]
   but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

3. Assume that $\varphi$ is a continuous real function on $(a,b)$ such that
   \[ \varphi \left( \frac{x+y}{2} \right) \leq \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(y) \]
   for all $x$ and $y$ in $(a,b)$. Prove that $\varphi$ is convex.

4. Suppose that $H$ is a Hilbert space with inner product $(\cdot,\cdot)$. Prove that if $L$ is a continuous linear functional on $H$, then there exists a unique $y \in H$ such that $Lx = (x,y)$.

5. State and prove the Banach-Steinhaus Theorem. [You may assume Baire’s Theorem in your proof.]

6. Prove that the total variation $|\mu|$ of a complex measure $\mu$ on $\mathcal{M}$ is a positive measure on $\mathcal{M}$.

7. Suppose that $f$ is an absolutely continuous, nondecreasing function on $[a,b]$. Prove that if $E$ is Lebesgue measurable with $m(E) = 0$, then $f(E)$ is Lebesgue measurable with $m(f(E)) = 0$.

8. Let $\mathcal{B}_k$ denote the $\sigma$-algebra of all Borel sets in $\mathbb{R}^k$. Prove that $\mathcal{B}_{m+n} = \mathcal{B}_m \times \mathcal{B}_n$.

9. Suppose that $A$ and $B$ are Lebesgue measurable subsets of $\mathbb{R}$, each having finite positive Lebesgue measure. Prove that $\chi_A \ast \chi_B$ is continuous and not identically equal to 0.

10. Suppose that $\{f_n\}$ is a uniformly bounded sequence of holomorphic functions on the region $\Omega$ such that $\{f_n(z)\}$ converges for every $z \in \Omega$. Prove that the convergence is uniform on every compact subset of $\Omega$. 