Instructions:

- All vector spaces on this exam are real.
- All measures on this exam are non-negative (i.e., $\mu(A) \in [0, \infty]$ for every measure $\mu$, and for every $A$ in the domain of $\mu$).
- Full credit on a problem will only be awarded for a complete solution.
- Submit solutions to no more than 7 of the 10 problems. (If you submit more than 7 solutions, your score will be based only on the first 7 solutions you submit.)
1. Let \((X, \mathcal{A}, \mu)\) be a measure space. Prove that \(\mu\) is \(\sigma\)-finite if and only if there is a \(\mu\)-integrable function \(f : X \to (0, \infty)\).

2. Let \((X, \mathcal{A}, \mu)\) be a measure space. Prove that if \(f_n \to 0\) in \(L^1(\mu)\) then \(f_n \to 0\) in measure; give a specific counterexample \(i.e.,\) choose \(X, \mathcal{A}, \mu,\) and \(f_n\) to show that the converse doesn’t hold.

3. Prove that no \(\sigma\)-algebra is countably infinite.

4. Prove that if \(f : [0, 1] \to \mathbb{R}\) has bounded variation, then there are nondecreasing functions \(f_1 : [0, 1] \to \mathbb{R}\) and \(f_2 : [0, 1] \to \mathbb{R}\) such that \(f = f_1 - f_2\).

5. Let \((X, \mathcal{A}, \mu)\) be a measure space, with \(\mu(X) < \infty\). For every \(B \subseteq X\), define
   \[
   \nu(B) := \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathcal{A} \text{ and } B \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.
   \]
   Prove that \(\mu(X) \leq \nu(B) + \nu(X \setminus B)\) for every \(B \subseteq X\).

6. Prove that if \(V\) is a finite-dimensional vector space then all norms on \(V\) are equivalent.

7. Let \((V, \|\cdot\|)\) be a normed vector space, let \(V^*\) be the set of bounded linear real-valued functions defined on \(V\), and let \(\|\cdot\|_*\) be the operator norm defined by
   \[
   \|f\|_* := \sup\{ |f(v)| : \|v\| \leq 1 \}.
   \]
   Prove that \((V^*, \|\cdot\|_*)\) is complete.

8. Define what a Hamel basis is and what a Schauder basis is. Is every Hamel basis a Schauder basis? Is every Schauder basis a Hamel basis? Explain.

9. Let \(H\) be a Hilbert space, let \(T : H \to H\) be linear and bounded, and let \(T^* : H \to H\) be the adjoint of \(T\).
   (a) Prove that the operator norm of \(T^*\) is no bigger than the operator norm of \(T\).
   (b) Prove that if the operator norm of \(T\) is 1, and \(h_0 \in H\) satisfies \(T(h_0) = h_0\), then \(T^*(h_0) = h_0\).

10. Let \((V, \|\cdot\|)\) be a normed vector space. We say a series \(\sum_i v_i\) in \(V\) is absolutely convergent if \(\sum_i \|v_i\|\) is convergent. Prove that \((V, \|\cdot\|)\) is complete if and only if every absolutely convergent series in \(V\) is convergent.