JANUARY 2014
ANALYSIS
PH.D. QUALIFYING EXAMINATION

Instructions:

- All vector spaces on this exam are real.
- All measures on this exam are non-negative (*i.e.*, $\mu(A) \in [0, \infty]$ for every measure μ, and for every A in the domain of μ).
- Full credit on a problem will only be awarded for a **complete** solution.
- Submit solutions to no more than 7 of the 10 problems. (If you submit more than 7 solutions, your score will be based only on the first 7 solutions you submit.)

1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that μ is σ-finite if and only if there is a μ-integrable function $f : X \to (0, \infty)$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that if $f_n \to 0$ in $L^1(\mu)$ then $f_n \to 0$ in measure; give a specific counterexample (*i.e.*, choose X, $\mathcal{A}$, μ, and f_n) to show that the converse doesn't hold.

3. Prove that no σ-algebra is countably infinite.

4. Prove that if $f : [0, 1] \to \mathbb{R}$ has bounded variation, then there are nondecreasing functions $f_1 : [0, 1] \to \mathbb{R}$ and $f_2 : [0, 1] \to \mathbb{R}$ such that $f = f_1 - f_2$.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) < \infty$. For every $B \subseteq X$, define
$$\nu(B) := \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathcal{A} \text{ and } B \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$
Prove that $\mu(X) \leq \nu(B) + \nu(X \setminus B)$ for every $B \subseteq X$.

6. Prove that if V is a finite-dimensional vector space then all norms on V are equivalent.

7. Let $(\mathsf{V}, \|\cdot\|)$ be a normed vector space, let V^* be the set of bounded linear real-valued functions defined on V, and let $\|\cdot\|_*$ be the operator norm defined by
$$\|f\|_* := \sup\{|f(v)| : \|v\| \leq 1\}.$$
Prove that $(\mathsf{V}^*, \|\cdot\|_*)$ is complete.

8. Define what a Hamel basis is and what a Schauder basis is. Is every Hamel basis a Schauder basis? Is every Schauder basis a Hamel basis? Explain.

9. Let H be a Hilbert space, let $\mathsf{T} : \mathsf{H} \to \mathsf{H}$ be linear and bounded, and let $\mathsf{T}^* : \mathsf{H} \to \mathsf{H}$ be the adjoint of T.
   (a) Prove that the operator norm of T^* is no bigger than the operator norm of T.
   (b) Prove that if the operator norm of T is 1, and $h_0 \in \mathsf{H}$ satisfies $\mathsf{T}(h_0) = h_0$, then $\mathsf{T}^*(h_0) = h_0$.

10. Let $(\mathsf{V}, \|\cdot\|)$ be a normed vector space. We say a series $\sum_i v_i$ in V is *absolutely convergent* if $\sum_i \|v_i\|$ is convergent. Prove that $(\mathsf{V}, \|\cdot\|)$ is complete if and only if every absoutely convergent series in V is convergent.