Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Spring 2011

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. Lip α – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
17. $f * g$ – the convolution of f and g: $(f * g)(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dx$ – the Fourier transform

Questions

1. Prove or disprove: If $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

2. Construct a sequence of continuous functions f_n on $[0,1]$ with $0 \le f_n \le 1$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = 0$$

but such that the sequence $\{f_n(x)\}$ does not converge for any $x \in [0,1]$.

3. Prove that if f and g are measurable functions on X with range in $[0,\infty]$ and $1 < p < \infty$, then

$$\int_X fg \, d\mu \le \left\{ \int_X f^p \, d\mu \right\}^{1/p} \left\{ \int_X g^q \, d\mu \right\}^{1/q}.$$

4. Prove that every nonempty, closed, convex subset E of a Hilbert space H contains a unique element of smallest norm. [You may assume that the norm on H is a continuous function.]

5. State and prove the Banach-Steinhaus Theorem. [You may assume Baire's Theorem in your proof.]

6. Suppose that μ is a finite positive measure on X and Φ is a bounded linear functional on $L^1(\mu)$. Prove that there exists a unique $g \in L^\infty(\mu)$ such that

$$\Phi(f) = \int_X fg \, d\mu \text{ with } \|\Phi\| = \|g\|_\infty.$$

[You may assume the Radon-Nikodym Theorem in your proof.]

7. If f is absolutely continuous on $[a,b]$ with $f' \in L^p$ for $1 < p < \infty$, then prove that $f \in \mathrm{Lip}(\alpha)$ for $\alpha = 1/q$.

8. Give an example of a function $f(x,y)$, for $x, y \in [0,1]$, where the conclusion of Fubini's Theorem fails to hold. Explain why the conclusion of Fubini's Theorem fails to hold for your example.

9. Prove that if $f \in L^1$, then $\hat{f} \in C_0$ and $\|\hat{f}\|_\infty \le \|f\|_1$. [You may assume the uniform continuity of the mapping $y \to f(x-y)$ of $\mathbb{R}$ to $L^1(\mathbb{R})$.]

10. Suppose μ is a complex measure on a measurable space X, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z}, \quad z \in \Omega.$$

Prove that f is representable by power series in Ω.