Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Spring 2012

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\mathcal{B}_X$ – the Borel σ-algebra in X
4. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
5. a.e.$[\mu]$ – almost every with respect to the measure μ
6. m – Lebesgue measure on $\mathbb{R}^k$
7. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
8. $\|f\|_\infty$ – the essential supremum of f
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dm(y)$
17. $C_c(X)$ – the continuous complex functions on X with compact support
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dm(x)$ – the Fourier transform

Questions

1. State and prove Fatou's Lemma. [You may assume Lebesgue's Monotone Convergence Theorem in your proof.]

2. For a locally compact Hausdorff space X, the Riesz Representation Theorem asserts that associated to a positive linear functional Λ on $C_c(X)$ there is a σ-algebra $\mathcal{M}$ in X satisfying $\mathcal{B}_X \subset \mathcal{M}$, and a unique positive measure μ on $\mathcal{M}$ such that

$$\Lambda f = \int_X f \, d\mu \text{ for every } f \in C_c(X).$$

List the remaining properties that $\mathcal{M}$ and μ satisfy, **AND** using some of these properties, prove the uniqueness of μ. [You may assume Urysohn's Lemma in your proof.]

3. For a positive measure μ and $0 < p < q < r \leq \infty$, prove that $L^p(\mu) \cap L^r(\mu) \subset L^q(\mu)$.

4. Find the unique real values of a, b, and c for which

$$\int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

is minimized. [Hint: the integral is $\|x^3 - a - bx - cx^2\|_2^2$ for the norm $\|\cdot\|_2 = \sqrt{(\cdot,\cdot)}$ on the real Hilbert space $L^2[-1,1]$ with inner product $(f,g) = \int_{-1}^{1} fg \, dm(x)$.]

5. Construct a bounded linear functional on some subspace of $L^1(\mathbb{R})$ which has two distinct norm-preserving extensions to $L^1(\mathbb{R})$.

6. Prove that if μ is a complex measure on a σ-algebra $\mathcal{M}$ in X, then there is an $h \in L^1(|\mu|)$ with $|h(x)| = 1$ for all $x \in X$ such that $d\mu = hd|\mu|$. [You may assume the Radon-Nikodym Theorem in your proof.]

7. Assume for a nondecreasing function $f : [a,b] \to \mathbb{R}$ that there is a positive Borel measure μ on $[a,b]$ for which $f(x) - f(a) = \mu([a,x))$, $x \in [a,b]$. Prove that $f'(x)$ exists a.e.$[m]$ and $f' \in L^1(m)$.

8. Prove that if $f, g \in L^1(\mathbb{R})$, then $h = f * g \in L^1(\mathbb{R})$ and $\|h\|_1 \leq \|f\|_1 \|g\|_1$. [You may assume Fubini's Theorem in your proof.]

9. Prove that if $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$. [You may assume that the map $y \to f(x - y)$ for fixed x is a uniformly continuous mapping of $\mathbb{R}$ into $L^1(\mathbb{R})$ in your proof.]

10. Prove that if μ is a finite Radon measure and if f is a positive function in $L^1(\mu)$, then the Borel measure ν defined by

$$\nu(E) = \int_E f \, d\mu, \quad E \in \mathcal{B}_X,$$

is a finite Radon measure. [Hint: if $g \in L^1(\mu)$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int_A |g| \, d\mu < \epsilon$ whenever $\mu(A) < \delta$.]