Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Spring 2016

Instructions: Answer exactly 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean $k$-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $B_X$ – the Borel $\sigma$-algebra in $X$
4. $(X, M, \mu)$ – a measure space where $X$ is a set, $M$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a measure on $M$
5. a.e.$[\mu]$ – almost every with respect to the measure $\mu$
6. $m$ – Lebesgue measure on $\mathbb{R}^k$
7. $\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$ – the $L^p$-norm of a $\mu$-measurable function $f: X \to \mathbb{C}$
8. $\|f\|_\infty$ – the essential supremum of $f$
9. $p, q$ – conjugate exponents where $\frac{1}{p} + \frac{1}{q} = 1$
10. $L^p(\mu)$ – the space of $\mu$-measurable functions $f: X \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f: \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
12. $\|\Gamma\| = \sup\{\|\Gamma x\|: x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma: X \to Y$ where $X$ and $Y$ are normed linear spaces
13. $|\lambda|$ – the total variation of a measure $\lambda$.
14. $\lambda \ll \mu$ – the measure $\lambda$ is absolutely continuous with respect to the measure $\mu$
15. $\lambda \perp \mu$ – the measures $\lambda$ and $\mu$ are mutually singular
16. $\frac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ where $\lambda \ll \mu$
17. Lip$\alpha$ – the space of complex functions $f$ on $[a, b]$ for which $\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty$; here $0 < \alpha \leq 1$
18. $f * g$ – the convolution of $f$ and $g$: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dm(y)$
19. $C_c(X)$ – the continuous complex functions on $X$ with compact support
20. $C_0(X)$ – the continuous complex functions on a LCH space $X$ which vanish at infinity
21. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dm(x)$ – the Fourier transform
Questions

1. State and prove Lebesgue’s Dominated Convergence Theorem.

2. Suppose \( \{f_k\}_{k=1}^\infty \) is a sequence of measurable functions defined on a measurable set \( E \subset X \) with \( \mu(E) < \infty \), and \( f_k \to f \) a.e. Prove that, for each \( \epsilon > 0 \), there is a set \( A_\epsilon \) with \( A_\epsilon \subset E, \mu(E - A_\epsilon) < \epsilon \) such that \( f_k \to f \) uniformly on \( A_\epsilon \).

3. Suppose \( f \) is a complex measurable function on \( X \), \( \mu \) is a positive measure on \( X \), and \( \varphi(p) = \int_X |f|^p \, d\mu = \|f\|^p_p \quad (0 < p < \infty) \).

   If \( r < p < s \), prove that \( \|f\|_p \leq \max(\|f\|_r,\|f\|_s) \).

4. Let \( \ell \) be a continuous linear functional on a Hilbert space \( H \). Prove that there exists a unique element \( g \in H \) such that \( \ell(f) = \langle f, g \rangle_H \) for all \( f \in H \), and that \( \|\ell\| = \|g\|_H \).

5. Let \( \mu \) be a positive measure on \( X \) with \( \mu(X) < \infty \), and \( 1 \leq p < \infty \).

   (a) If \( f_n \in L^p(\mu) \) and \( \|f_n - f\|_p \to 0 \), prove that \( f_n \to f \) in measure.

   (b) Conversely if \( f_n \to f \) in measure, prove that \( \{f_n\} \) has a subsequence which converges to \( f \) a.e. \( [\mu] \).

6. Let \( f \) be a continuous function on a compact metric space \( X \). Prove that \( f \) is uniformly continuous on \( X \).

7. If the functions \( f_n(z) \) are analytic and \( \neq 0 \) in a region \( \Omega \), and if \( f_n(z) \) converges to \( f(z) \), uniformly on every compact subset of \( \Omega \), prove that \( f(z) \) is either identically zero or never equal to zero in \( \Omega \).

8. If \( f(z) \) is analytic and \( \text{Im} f(z) \geq 0 \) for \( \text{Im} \ z > 0 \), show that

\[
\left| \frac{f(z) - f(z_0)}{f(z) - f(z_0)} \right| \leq \frac{|z - z_0|}{|z - z_0|}.
\]

9. Compute

\[
\int_0^\infty \cos(x^2) \, dx.
\]

10. Let \( f \) be analytic in a region \( \Omega \). If \( a \) is a point in \( \Omega \) such that \( f^{(n)}(a) = 0 \) for \( n = 0, 1, 2, \ldots \), prove that \( f(z) = 0 \) for all \( z \in \Omega \).