Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Spring 2016

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\mathcal{B}_X$ – the Borel σ-algebra in X
4. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
5. a.e.$[\mu]$ – almost every with respect to the measure μ
6. m – Lebesgue measure on $\mathbb{R}^k$
7. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f \colon X \to \mathbb{C}$
8. $\|f\|_\infty$ – the essential supremum of f
9. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
10. $L^p(\mu)$ – the space of μ-measurable functions $f \colon X \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f \colon \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
12. $\|\Gamma\| = \sup\{\|\Gamma x\| \colon x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma \colon X \to Y$ where X and Y are normed linear spaces
13. $|\lambda|$ – the total variation of a measure λ.
14. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
15. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
16. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
17. $\operatorname{Lip} \alpha$ – the space of complex functions f on $[a, b]$ for which $\displaystyle\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
18. $f * g$ – the convolution of f and g: $(f * g)(x) = \displaystyle\int_{-\infty}^{\infty} f(x - y) g(y) \, dm(y)$
19. $C_c(X)$ – the continuous complex functions on X with compact support
20. $C_0(X)$ – the continuous complex functions on a LCH space X which vanish at infinity
21. $\hat{f}(t) = \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform

Questions

1. State and prove Lebesgue's Dominated Convergence Theorem.

2. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set $E \subset X$ with $\mu(E) < \infty$, and $f_k \to f$ a.e. Prove that, for each $\epsilon > 0$, there is a set A_ϵ with $A_\epsilon \subset E$, $\mu(E - A_\epsilon) < \epsilon$ such that $f_k \to f$ uniformly on A_ϵ.

3. Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \qquad (0 < p < \infty).$$

   If $r < p < s$, prove that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$.

4. Let ℓ be a continuous linear functional on a Hilbert space H. Prove that there exists a unique element $g \in H$ such that $\ell(f) = \langle f, g \rangle_H$ for all $f \in H$, and that $\|\ell\| = \|g\|_H$.

5. Let μ be a positive measure on X with $\mu(X) < \infty$, and $1 \leq p < \infty$.
   (a) If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, prove that $f_n \to f$ in measure.
   (b) Conversely if $f_n \to f$ in measure, prove that $\{f_n\}$ has a subsequence which converges to f a.e. $[\mu]$.

6. Let f be a continuous function on a compact metric space X. Prove that f is uniformly continuous on X.

7. If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, prove that $f(z)$ is either identically zero or never equal to zero in Ω.

8. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \left| \frac{z - z_0}{z - \bar{z}_0} \right|.$$

9. Compute

$$\int_0^\infty \cos(x^2) \, dx.$$

10. Let f be analytic in a region Ω. If a is a point in Ω such that $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \ldots$, prove that $f(z) = 0$ for all $z \in \Omega$.