Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2010

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. Lip α – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
17. $f * g$ – the convolution of f and g: $(f * g)(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - y)g(y) \, dy$
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dx$ – the Fourier transform

Questions

1. Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$. Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$, or give a counterexample.

2. Construct a sequence of continuous functions f_n on $[0, 1]$ such that $0 \leq f_n \leq 1$, such that $\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = 0$, but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0, 1]$.

3. State and prove Jensen's Inequality.

4. Prove that every orthonormal basis of a separable Hilbert space is countable.

5. Prove that if X and Y are Banach spaces and if Λ is a bounded linear transformation of X onto Y which is also one-to-one, then Λ^{-1} is a bounded linear transformation of Y onto X. [You may assume the Open Mapping Theorem in your proof.]

6. For a complex measure μ on a σ-algebra $\mathcal{M}$ in X, prove that there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and $d\mu = h d|\mu|$. [You may assume the Radon-Nikodym Theorem in your proof.]

7. Give the definition of a Lebesgue point for $f \in L^1(\mathbb{R}^k)$. State what it means for a sequence $\{E_i\}$ of Borel sets in $\mathbb{R}^k$ to shrink nicely to a point $x \in \mathbb{R}^k$. Prove for each $f \in L^1(\mathbb{R}^k)$ and any sequence $\{E_i\}$ of Borel sets in $\mathbb{R}^k$ that shrink nicely to a Lebesgue point x of f, that

$$f(x) = \lim_{i\to\infty} \frac{1}{m(E_i(x))} \int_{E_i(x)} f \, dm.$$

Prove that if $f \in L^1(\mathbb{R}^1)$, and $F(x) = \int_{-\infty}^{x} f \, dm$ for $-\infty < x < \infty$, then $F'(x) = f(x)$ at every Lebesgue point of f.

8. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be σ-finite measure spaces, and let f be an $\mathcal{S}\times\mathcal{T}$-measurable function on $X \times Y$. Prove that if $0 \leq f \leq \infty$, then $\phi(x) = \int_Y f(x, y) \, d\lambda$ is $\mathcal{S}$-measurable, $\psi(y) = \int_X f(x, y) \, d\mu$ is $\mathcal{T}$-measurable, and

$$\int_X \phi \, d\mu = \int_{X\times Y} f \, d(\mu \times \lambda) = \int_Y \psi \, d\lambda.$$

9. Prove that if $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$. [You may assume in your proof that the translation $y \to f(x - y)$ for $x, y \in \mathbb{R}$ and $f \in L^1(\mathbb{R})$ is a uniformly continuous mapping of $\mathbb{R}$ into $L^1(\mathbb{R})$.]

10. Prove that if μ is a complex measure on a measurable space X, if ϕ is a complex measurable function on X, and if Ω is an open set in the complex plane which does not intersect $\phi(X)$, then the function

$$f(z) = \int_X \frac{d\mu(\zeta)}{\phi(\zeta) - z}, \quad z \in \Omega,$$

is representable by power series in Ω.