Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2011

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. Lip α – the space of complex functions f on $[a, b]$ for which $\sup\limits_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
17. $f * g$ – the convolution of f and g: $(f * g)(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ixt} \, dx$ – the Fourier transform

Questions

1. State and prove Lebesgue's Dominated Convergence Theorem. [You may assume Fatou's Lemma in your proof.]

2. Prove that every outer regular Borel measure on a locally compact, σ-compact Hausdorff space X is inner regular.

3. Prove for every positive measure μ and every $1 \leq p \leq \infty$, that $L^p(\mu)$ is a complete metric space.

4. Prove that the dual of $L^2(\mu)$ is $L^2(\mu)$.

5. Construct a bounded linear functional on some subspace of $L^1(\mu)$ which has at least two distinct norm-preserving extensions to $L^1(\mu)$.

6. State and proof the Hahn Decomposition Theorem. [You may assume the Radon-Nikodym Theorem in your proof.]

7. Let $f \in L^1(\mathbb{R}^k)$. Let $B_r(x)$ be the ball of radius $r \geq 0$ and center $x \in \mathbb{R}^k$. Prove that for a.e.$[m]$-$x \in \mathbb{R}^k$, there holds

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \, dm(y) = 0.$$

[You may assume the weak L^1 estimate on the maximal function.]

8. Prove that if $f, g \in L^1(\mathbb{R})$, then $h = f * g$ is $L^1(\mathbb{R})$ and $\|h\|_1 \leq \|f\|_1 \, \|g\|_1$. [You may assume Fubini's Theorem in your proof.]

9. Let g be the characteristic function on $[-2, 2]$ and h the characteristic function on $[-1, 1]$. Find a function $f \in L^1(\mathbb{R})$ for which $g * h$ is the Fourier transform of f.

10. Let α be a complex number such that $|\alpha| \neq 1$. Use the method of residues to compute

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha \cos\theta + \alpha^2} \, d\theta.$$