Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2011

Instructions: Answer exactly 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.
1. \( \mathbb{R}^k \) – Euclidean \( k \)-dimensional space
2. \( \mathbb{C} \) – the complex numbers
3. \((X, \mathcal{M}, \mu)\) – a measure space where \( X \) is a set, \( \mathcal{M} \) is a \( \sigma \)-algebra of subsets of \( X \), and \( \mu \) is a measure on \( \mathcal{M} \)
4. a.e.\([\mu]\) – almost every with respect to the measure \( \mu \)
5. \( m \) – Lebesgue measure on \( \mathbb{R}^k \)
6. \( \|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p} \) – the \( L^p \)-norm of a \( \mu \)-measurable function \( f : X \to \mathbb{C} \)
7. \( \|f\|_\infty \) – the essential supremum of \( f \)
8. \( p, q \) – conjugate exponents where \( \frac{1}{p} + \frac{1}{q} = 1 \)
9. \( L^p(\mu) \) – the space of \( \mu \)-measurable functions \( f : X \to \mathbb{C} \) with \( \|f\|_p < \infty \)
10. \( L^p(\mathbb{R}^k) \) – the space of Lebesgue measurable functions \( f : \mathbb{R}^k \to \mathbb{C} \) with \( \|f\|_p < \infty \)
11. \( \|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\} \) – operator norm of a linear transformation \( \Gamma : X \to Y \) where \( X \) and \( Y \) are normed linear spaces
12. \(|\lambda|\) – the total variation of a measure \( \lambda \).
13. \( \lambda \ll \mu \) – the measure \( \lambda \) is absolutely continuous with respect to the measure \( \mu \)
14. \( \lambda \perp \mu \) – the measures \( \lambda \) and \( \mu \) are mutually singular
15. \( \frac{d\lambda}{d\mu} \) – the Radon-Nikodym derivative of \( \lambda \) with respect to \( \mu \) where \( \lambda \ll \mu \)
16. \( \text{Lip } \alpha \) – the space of complex functions \( f \) on \([a, b]\) for which \( \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \); here \( 0 < \alpha \leq 1 \)
17. \( f \ast g \) – the convolution of \( f \) and \( g \): \((f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy \)
18. \( C_0(\mathbb{R}^k) \) – the continuous complex functions on \( \mathbb{R}^k \) which vanish at infinity
19. \( \hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dx \) – the Fourier transform
Questions

1. State and prove Lebesgue’s Dominated Convergence Theorem. [You may assume Fatou’s Lemma in your proof.]

2. Prove that every outer regular Borel measure on a locally compact, \(\sigma\)-compact Hausdorff space \(X\) is inner regular.

3. Prove for every positive measure \(\mu\) and every \(1 \leq p \leq \infty\), that \(L^p(\mu)\) is a complete metric space.

4. Prove that the dual of \(L^2(\mu)\) is \(L^2(\mu)\).

5. Construct a bounded linear functional on some subspace of \(L^1(\mu)\) which has at least two distinct norm-preserving extensions to \(L^1(\mu)\).

6. State and proof the Hahn Decomposition Theorem. [You may assume the Radon-Nikodym Theorem in your proof.]

7. Let \(f \in L^1(\mathbb{R}^k)\). Let \(B_r(x)\) be the ball of radius \(r \geq 0\) and center \(x \in \mathbb{R}^k\). Prove that for a.e.\([m]\)-\(x \in \mathbb{R}^k\), there holds \[
\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \, dm(y) = 0.
\]
[You may assume the weak \(L^1\) estimate on the maximal function.]

8. Prove that if \(f, g \in L^1(\mathbb{R})\), then \(h = f \ast g\) is \(L^1(\mathbb{R})\) and \(\|h\|_1 \leq \|f\|_1 \|g\|_1\). [You may assume Fubini’s Theorem in your proof.]

9. Let \(g\) be the characteristic function on \([-2, 2]\) and \(h\) the characteristic function on \([-1, 1]\). Find a function \(f \in L^1(\mathbb{R})\) for which \(g \ast h\) is the Fourier transform of \(f\).

10. Let \(\alpha\) be a complex number such that \(|\alpha| \neq 1\). Use the method of residues to compute \[
\int_0^{2\pi} \frac{1}{1 - 2\alpha \cos \theta + \alpha^2} \, d\theta.
\]