Ph.D. QUALIFIER EXAMINATION: ANALYSIS

January 2017

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\Im z$ is the imaginary part of z.
4. $\mathcal{B}_X$ – the Borel σ-algebra in X
5. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
6. a.e.$[\mu]$ – almost every with respect to the measure μ
7. m – Lebesgue measure on $\mathbb{R}^k$
8. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f \colon X \to \mathbb{C}$
9. $\|f\|_\infty$ – the essential supremum of f
10. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
11. $L^p(\mu)$ – the space of μ-measurable functions $f \colon X \to \mathbb{C}$ with $\|f\|_p < \infty$
12. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f \colon \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
13. $\|\Gamma\| = \sup\{\|\Gamma x\| \colon x \in X, \|x\| \le 1\}$ – operator norm of a linear transformation $\Gamma \colon X \to Y$ where X and Y are normed linear spaces
14. $|\lambda|$ – the total variation of a measure λ.
15. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
16. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
17. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
18. $\operatorname{Lip} \alpha$ – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \le 1$
19. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dm(y)$
20. $C_c(X)$ – the continuous complex functions on X with compact support
21. $C_0(X)$ – the continuous complex functions on a LCH space X which vanish at infinity
22. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform.

Questions

1. Let

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

Show that $f(x, y) = \operatorname{sgn}(x - y)e^{-|x-y|}$ satisfies

$$\int_0^\infty dx \int_0^\infty f(x, y)dy = -\int_0^\infty dy \int_0^\infty f(x, y)dx = -1.$$

Why does this not contradict Fubini's theorem?

2. Suppose μ is a complex measure on $[0, 2\pi)$. Let

$$\hat{\mu}(n) = \int_{[0,2\pi)} e^{-int} d\mu(t) \quad (n \in \mathbb{Z}).$$

Assume that $\hat{\mu}(n) \to 0$ as $n \to +\infty$. Prove that $\hat{\mu}(n) \to 0$ as $n \to -\infty$.

Hint: The assumption holds with $f d\mu$ in place of $d\mu$ if f is any trigonometric polynomial, hence if f is continuous, hence if f is any bounded Borel function, hence if $d\mu$ is replaced by $d|\mu|$.

3. Let K be a compact topological space and let $f_0, f_1, f_2, \dots$ be continuous real-valued functions on K. Suppose for every $x \in K$ the sequence $(f_n(x))_{n=1}^\infty$ is monotone increasing and converges pointwise to $f_0(x)$. Show that $(f_n)_{n=1}^\infty$ converges uniformly to f_0.

4. Show that the result in Question 3 need not hold if only two of the following conditions hold: (i) K is compact; (ii) the functions are continuous; (iii) $(f_n(x))_{n=1}^\infty$ is monotonic and converges pointwise to $f_0(x)$. (Thus three 'counterexamples' are required.)

5. (i) Under what conditions can the integral around a closed curve C,

$$\int_C \frac{f'(z)}{f(z)}\, dz$$

(f meromorphic in a region containing C) be expressed using certain counts of zeros and poles of f? Give a precise statement.

(ii) Suppose f_n is holomorphic in a region Ω and f_n has no zeros in Ω. Suppose $f_n \to f$ uniformly on compact subsets of Ω where f is not identically zero in Ω. Show that

$$\int_C \frac{f'(z)}{f(z)}\, dz = 0$$

for a sufficiently small circle around any point z_0 in Ω. Deduce that f has no zeros in Ω.

6. Show that the mean value formula for harmonic functions,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

remains valid for $u = \log|1 + z|$, $z_0 = 0$, $r = 1$. Use this fact to compute

$$\int_0^{\pi} \log \sin \theta \, d\theta.$$

7. State and prove the Riemann-Lebesgue lemma for a function $f \in L^1(\mathbb{R})$. (You may assume standard density results for $L^1(\mathbb{R})$.)

8. State the Riemann mapping in a form that makes the mapping functions unique.

9. Let L be a continuous linear functional on a Hilbert space H. Prove that there is a unique $y \in H$ such that

$$Lx = (x, y) \quad (x \in H).$$

(You may assume the decomposition of H using M, $M^{\perp}$ whenever M is a closed subspace.)

10. (i) Suppose that λ, μ, ν are complex measures on a σ-algebra $\mathcal{M}$. If $\lambda \ll |\mu|$ and $\mu \perp \nu$, show that $\lambda \perp \nu$.

(ii) For two complex measures λ, μ on a σ-algebra $\mathcal{M}$, define $\lambda \sim \mu$ to mean

$$\lambda \ll |\mu| \text{ and } \mu \ll |\lambda|.$$

Show that '$\sim$' is an equivalence relation on the set M of complex measures on $\mathcal{M}$, and find the equivalence classes.