Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2015

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.
Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\mathcal{B}_X$ – the Borel σ-algebra in X
4. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
5. a.e.$[\mu]$ – almost every with respect to the measure μ
6. m – Lebesgue measure on $\mathbb{R}^k$
7. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
8. $\|f\|_\infty$ – the essential supremum of f
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dm(y)$
17. $C_c(X)$ – the continuous complex functions on X with compact support
18. $C_0(X)$ – the continuous complex functions on a LCH space X which vanish at infinity
19. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform

Questions

1. For a positive measure μ on a σ-algebra $\mathcal{M}$ in X, suppose that $\{f_n\}$ is a sequence of $L^1(\mu)$ functions converging point-wise to $f \in L^1(\mu)$. Prove that if $\|f_n\|_1 \to \|f\|_1$ as $n \to \infty$, then for every $E \in \mathcal{M}$,

$$\lim_{n\to\infty} \int_E |f_n| \, d\mu = \int_E |f| \, d\mu.$$

2. If $0 < p < q < r \leq \infty$, then $L^p(\mu) \cap L^r(\mu) \subset L^q(\mu)$ and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ where $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$.

3. Let X and Y be normed linear spaces and $L(X, Y)$ be the space of bounded linear transformations from X to Y. Prove that if Y is a Banach space then $L(X, Y)$, equipped with the operator norm, is a Banach space. [Note: X is not assumed to be a Banach space.]

4. The Riesz Representation Theorem for a Hilbert space H with inner product $(\cdot, \cdot)$ states that if L is a continuous linear functional on H, then there exists a unique $y \in H$ such that for all $x \in H$, $Lx = (x, y)$. Prove that $\|L\| = \|y\|$.

5. Prove that

$$\lim_{n\to\infty} \int_E \cos nx \, dx = 0$$

for any Lebesgue measurable subset E of $[0, 2\pi]$.

6. Suppose $f, g \in L^1(\mathbb{R})$. Prove that the function $h(x, y) = f(x-y)g(y)$ is Borel-measurable on $\mathbb{R}^2$.

7. Let μ be a complex measure on a σ-algebra $\mathcal{M}$ in X. Prove for every $E \in \mathcal{M}$ that

$$|\mu|(E) = \sup \left\{ \left| \int_E f d\mu \right| : f \text{ is measurable and } |f| \leq 1 \right\}.$$

8. Let X be a locally compact Hausdorff space and $M(X)$ the set of all regular complex Borel measures on X. If $\mu, \nu \in M(X)$, prove that $\mu - \nu \in M(X)$.

9. Construct a sequence of continuous functions $\{f_n\}$ on $[0, 1]$ such that $0 \leq f_n \leq 1$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = 0,$$

but such that $f_n(x)$ does not go to 0 for any $x \in [0, 1]$.

10. Suppose μ is a positive measure on X satisfying $\mu(X) = 1$. A sequence $\{f_n\}$ of complex measurable functions on X is said to converge in measure to a measurable function f if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n > N$ there holds

$$\mu\big(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\big) < \epsilon.$$

For $1 \leq p < \infty$, prove that if $f_n \in L^p(\mu)$ for all n, $f \in L^p(\mu)$, and $\|f_n - f\|_p \to 0$, then f_n converges in measure to f.