MS Algebra Exam – August 2016

Answer questions 1–9. Then answer only one of 10, 11. Partial credit will be given.

1. If $G$ and $H$ are finite groups, and $G \times H$ is cyclic, prove that every subgroup of $G \times H$ is of the form $A \times B$ where $A$ is a subgroup of $G$, and $B$ is a subgroup of $H$.

2. Prove there is no homomorphism from the symmetric group $S_4$ onto the dihedral group $D_4$ (symmetries of the square).

3. Prove that the multiplicative group of positive rationals is isomorphic to the additive group of $\mathbb{Z}[x]$ (polynomials with integer coefficients).

4. Let $f : R \to S$ be a surjective homomorphism of commutative rings. If $J$ is a prime ideal in $S$ and $I = \{ r \in R | f(r) \in J \}$, prove that $I$ is a prime ideal in $R$.

5. Let $R = \{ a + b\sqrt{2} | a, b \in \mathbb{Z} \}$ and let $R'$ consist of all $2 \times 2$ matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$, for $a, b \in \mathbb{Z}$. Show $R$ and $R'$ are isomorphic rings. (Be sure to show they are rings.)

6. Let $\{ u_1, u_2, \ldots, u_n \}$ be an orthonormal basis for $\mathbb{R}^n$ (column vectors), and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be any real scalars. Define

$$ A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots + \lambda_n u_n u_n^T. $$

(a) Show that $A$ is symmetric.
(b) Show that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$.

7. Let $U$ be an $m \times n$ real valued matrix with orthonormal columns, and let $x \in \mathbb{R}^n$ (column vectors). Show that the norm $||Ux|| = ||x||$.

8. Prove that if $A$ is a diagonalizable matrix, then so is $A^n$ for every positive integer $n$.

9. If $F$ is a field of characteristic $p \neq 0$, show that $(a + b)^p = a^p + b^p$ for all $a, b \in F$.

Now answer only one of:

10. Let $F$ be the splitting field of $x^3 + 2$ over $\mathbb{Q}$. Prove there are exactly four intermediate (strictly) fields between $\mathbb{Q}$ and $F$. List them.

11. Let $V$ be an $FG$-module and let $W$ be an $FG$-submodule. Show that $V/W$ is an $FG$-module. Here $V/W$ is the set of cosets of $W$ in $V$ as abelian groups under addition.