MS Algebra Exam – August 2016

Answer questions 1–9. Then answer only one of 10, 11. Partial credit will be given.

1. If G and H are finite groups, and $G \times H$ is cyclic, prove that every subgroup of $G \times H$ is of the form $A \times B$ where A is a subgroup of G, and B is a subgroup of H.

2. Prove there is no homomorphism from the symmetric group S_4 onto the dihedral group D_4 (symmetries of the square).

3. Prove that the multiplicative group of positive rationals is isomorphic to the additive group of $\mathbb{Z}[x]$ (polynomials with integer coefficients).

4. Let $f : R \to S$ be a surjective homomorphism of commutative rings. If J is a prime ideal in S and $I = \{r \in R | f(r) \in J\}$, prove that I is a prime ideal in R.

5. Let $R = \left\{a + b\sqrt{2} | a,\ b \in \mathbb{Z}\right\}$ and let R' consist of all 2×2 matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$. for $a, b \in \mathbb{Z}$. Show R and R' are isomorphic rings. (Be sure to show they are rings.)

6. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ (column vectors), and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be any real scalars. Define
$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \ldots + \lambda_n \mathbf{u}_n \mathbf{u}_1^T.$$
   (a) Show that A is symmetric.
   (b) Show that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

7. Let U be an $m \times n$ real valued matrix with orthonormal columns, and let $\mathbf{x} \in \mathbb{R}^n$ (column vectors). Show that the norm $||U\mathbf{x}|| = ||\mathbf{x}||$.

8. Prove that if A is a diagonalizable matrix, then so is A^n for every positive integer n.

9. If F is a field of characteristic $p \neq 0$, show that $(a + b)^p = a^p + b^p$ for all $a, b \in F$.

Now answer only one of:

10. Let F be the splitting field of $x^3 + 2$ over $\mathbb{Q}$. Prove there are exactly four intermediate (strictly) fields between $\mathbb{Q}$ and F. List them.

11. Let V be an FG-module and let W be an FG-submodule. Show that V/W is an FG-module. Here V/W is the set of cosets of W in V as abelian groups under addition.