**MS Algebra Exam – August 2015**

*Answer questions* $1-9$. *Then answer only one of* $10, 11$. *Partial credit will be given.*

1. Let $A$ be a finite abelian group of order $n$. Let $p$ be a prime dividing $n$. Show that there is a subgroup $H$ of $A$ of order $p$.

2. How many elements are conjugate to $(1,2)(3,4,5,6)(7,8)$ in the symmetric group $S_8$?

3. Let $\mathbb{Z}$ be the group of integers (with additive notation) and put $G = \mathbb{Z} \times \mathbb{Z}$. Let $H$ be the subset of all elements of the form
$$(3n + 2m, 4n + 5m), \text{ where } n, m \in \mathbb{Z}.$$
Show that $H$ is a subgroup of $G$ and that $G/H \cong \mathbb{Z}/7\mathbb{Z}$.

4. If $P$ is a prime ideal in a commutative ring $R$, prove that the set $P \times P$ is an ideal in $R \times R$, but not a prime ideal.

5. If $F$ is a field, prove that $F[x]$ is an integral domain.

6. Prove that $x^4 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

7. Is the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizable? If so, diagonalize this matrix. If not, explain why not.

8. What is the dimension of the subspace $H$ of all $3 \times 3$ symmetric matrices with real entries which have diagonal entries all equal to zero? Give a basis for $H$.

9. Prove or find a counterexample to the following statement.

   *Matrices with equivalent characteristic polynomials are always similar.*

**Now answer only one of:**

10. Prove that the roots of $x^5 - 6x + 3 = 0$ cannot be written as radical expressions over the rationals.

11. Let $V$ be a finite-dimensional $\mathbb{C}G$-module, where $G$ is a finite group. If $U$ is a submodule of $V$, show that there is a submodule $W$ of $V$ such that $V = U \oplus W$.