Algebra Masters Exam, February 2015

Answer all questions. Partial credit will be given.

- 1. Let D_{2n} denote the dihedral group of order 2n. Show that $D_{12} \cong D_6 \times D_2$. (Here D_2 is just the cyclic group of order 2.)
- 2. Let $G = GL(2, \mathbb{F}_3)$, the group of invertible 2×2 matrices over the field with 3 elements. Show that G is not a simple group, i.e. show that G has a proper non-trivial normal subgroup.
- 3. Let G be an Abelian group with identity e, and let $n \ge 1$ be a fixed positive integer. Let $H = \{g^n : g \in G\}$ and let $K = \{g \in G : g^n = e\}$. Prove that $G/K \cong H$. Also, give a counter-example in the non-Abelian case.
- 4. Let P be a prime ideal in a commutative ring R. Prove that the set $P \times P$ is an ideal in $R \times R$. Is it a prime ideal? Justify your answer.
- 5. Let $m, n \ge 1$ be positive integers. Prove that there is an isomorphism of rings $\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ if and only if gcd(m, n) = 1.
- 6. Justify whether or not the matrix $\begin{pmatrix} 5 & 1 & -3 \\ 6 & -11 & 2 \\ -12 & 0 & 1 \end{pmatrix} \in \mathbb{M}_3(\mathbb{Q})$ is invertible. Is the answer the same if we think of this matrix as an element of $\mathbb{M}_3(\mathbb{Z})$?
- 7. Let V be a finite dimensional vector space over \mathbb{C} , and let $T: V \to V$ be a linear transformation. Prove that T has at least one eigenvector. Show that the same conclusion does not need to be true if V is infinite dimensional.
- 8. Prove that the quotient ring $\mathbb{R}[x]/(x^2-2)$ is isomorphic to $\mathbb{R} \times \mathbb{R}$. Is this a field? Justify your answer.
- 9. Explicitly compute $\Phi_{12}(x) \in \mathbb{Z}[x]$, the 12th cyclotomic polynomial. Factor Φ_{12} completely over \mathbb{Q} and over \mathbb{F}_2 .
- 10. Describe the Galois group of the splitting field over \mathbb{Q} for the polynomial $x^3 + 2 \in \mathbb{Z}[x]$, as a set of permutations of the roots.