Algebra Masters Exam, February 2015
Answer all questions. Partial credit will be given.

1. Let $D_{2n}$ denote the dihedral group of order $2n$. Show that $D_{12} \cong D_6 \times D_2$. (Here $D_2$ is just the cyclic group of order 2.)

2. Let $G = GL(2, F_3)$, the group of invertible $2 \times 2$ matrices over the field with 3 elements. Show that $G$ is not a simple group, i.e. show that $G$ has a proper non-trivial normal subgroup.

3. Let $G$ be an Abelian group with identity $e$, and let $n \geq 1$ be a fixed positive integer. Let $H = \{g^n : g \in G\}$ and let $K = \{g \in G : g^n = e\}$. Prove that $G/K \cong H$. Also, give a counter-example in the non-Abelian case.

4. Let $P$ be a prime ideal in a commutative ring $R$. Prove that the set $P \times P$ is an ideal in $R \times R$. Is it a prime ideal? Justify your answer.

5. Let $m, n \geq 1$ be positive integers. Prove that there is an isomorphism of rings $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.

6. Justify whether or not the matrix $\begin{pmatrix} 5 & 1 & -3 \\ 6 & -11 & 2 \\ -12 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{Q})$ is invertible. Is the answer the same if we think of this matrix as an element of $M_3(\mathbb{Z})$?

7. Let $V$ be a finite dimensional vector space over $\mathbb{C}$, and let $T : V \to V$ be a linear transformation. Prove that $T$ has at least one eigenvector. Show that the same conclusion does not need to be true if $V$ is infinite dimensional.

8. Prove that the quotient ring $\mathbb{R}[x]/(x^2 - 2)$ is isomorphic to $\mathbb{R} \times \mathbb{R}$. Is this a field? Justify your answer.

9. Explicitly compute $\Phi_{12}(x) \in \mathbb{Z}[x]$, the 12th cyclotomic polynomial. Factor $\Phi_{12}$ completely over $\mathbb{Q}$ and over $\mathbb{F}_2$.

10. Describe the Galois group of the splitting field over $\mathbb{Q}$ for the polynomial $x^3 + 2 \in \mathbb{Z}[x]$, as a set of permutations of the roots.