MS Algebra Exam – February 2017

Answer questions 1–9. Then answer only one of 10, 11. Partial credit will be given.
REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

1. Let G be a group of order 9. Prove that G is Abelian.

2. Let a and b be elements of a group. Prove that the elements ab and ba have the same order.

3. Let α and β be transpositions (2-cycles) in S_n, with $\alpha \neq \beta$. Let $\gamma = \alpha\beta$ and prove that γ is either a 3-cycle, or a product of two 3-cycles.

4. Let $S = \left\{ \frac{m}{2n+1} : m, n \in \mathbb{Z}, \gcd(m, 2n+1) = 1 \right\}$. Let $I = \left\{ \frac{m}{k} \in S : m \text{ is even } \right\}$. Prove that $S/I \cong \mathbb{Z}_2$.

5. Let R be a *commutative* ring with identity, and suppose that R has no ideals except 0 and R. Prove that R is a field. Explain why the theorem is false if R is *not commutative.*

6. Let A be an $m \times n$ matrix with real entries. Show Row $A = (\text{Nul } A)^{\perp}$.

7. Let A be a real $k \times k$ matrix with k distinct eigenvalues; where each eigenvalue is in the interval $(0, 1)$. Prove that $A^n \to (0)$ (the matrix of zeros) as $n \to \infty$.

8. A submatrix of A is any matrix that results from deleting some (or none) of the rows and/or columns of A. Prove that if the rank of A is r then A contains an invertible $r \times r$ submatrix.

9. Let K and F be fields. If $K \supset F$ is such that $[K : F] = p$, p a prime, show that $K = F(a)$ for every a in K that is not in F.

Now answer only one of:

10. Prove that the Galois group of $x^5 - 6x + 3$ over the rationals is S_5, so this polynomial cannot be solved by radicals.

11. Let $G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ (the dihedral group generated by elements a and b). Let H be the subgroup $\langle a^2, b \rangle$. Define U to be the 1-dimensional subspace of $\mathbb{C}H$ spanned by
$$1 - a^2 + b - a^2 b.$$
Show that U is a $\mathbb{C}H$–submodule of $\mathbb{C}H$.