

**Algebra Masters Exam, February 2014**

*Answer all questions. Partial credit will be given.*

- (1) Prove that subgroups and quotient groups of cyclic groups are cyclic.
- (2) Show that a group  $G$  of order 56 with a non-normal Sylow 7-subgroup has a normal Sylow 2-subgroup. Conclude that  $G$  is not a simple group.
- (3) Show that the symmetric group  $S_n$  is generated by the set of transpositions  $(i, j)$ .
- (4) How many left ideals are in the ring  $M_2(\mathbb{F}_3)$  of  $2 \times 2$  matrices over the field  $\mathbb{F}_3$ ? List them all.
- (5) Prove that the subring  $\mathbb{Z}[x^2, x^3]$  of the ring  $\mathbb{Z}[x]$  is not a U.F.D.
- (6) Prove that eigenvectors  $v_1, v_2$  belonging to distinct eigenvalues  $\lambda_1, \lambda_2$  of an  $n \times n$  matrix  $A$  with entries in a field, are linearly independent.
- (7) Let  $A = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 7 \end{pmatrix}$ . Are  $A$  and  $B$  similar? Explain.
- (8) Construct a field of order 16 and find an element of that field which has multiplicative order 5.
- (9) Let  $p$  be a prime integer. Describe the Galois group of the splitting field over  $\mathbb{Q}$  for the polynomial  $x^3 - p$ .