Algebra Masters Exam, January 2014
Answer all questions. Partial credit will be given.

1. Let G be a finite group with center $Z(G)$. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

2. Prove that the center of a finite p-group is nontrivial.

3. Consider the symmetric group S_n.
   (a) How many conjugates are there of $(1, 2)$ in S_n (for $n \geq 2$)?
   (b) How many conjugates are there of $(1, 2, 3)$ in S_n (for $n \geq 3$)?
   (c) How many conjugates are there of $(1, 2)(3, 4)$ in S_n (for $n \geq 4$)?

4. Let S be the ring $\mathbb{M}_2(\mathbb{R})$ of 2×2 matrices over the real numbers $\mathbb{R}$. Show that S has exactly 2 two-sided ideals; namely, (0) and S.

5. Find a subring of $\mathbb{M}_2(\mathbb{R})$ isomorphic to $\mathbb{C}$, and give an explicit isomorphism.

6. Show that a finite integral domain (with $1 \neq 0$) is a field.

7. Prove the Steinitz exchange lemma: If $\{v_1, \ldots, v_m\}$ are linearly independent vectors in a vector space V, and $\{w_1, \ldots, w_n\}$ span V, then $m \leq n$ and (after reordering the w_i if necessary) the set $\{v_1, \ldots, v_m, w_{m+1}, \ldots, w_n\}$ spans V.

8. Show how to construct $\sqrt{2 + \sqrt{2}}$ with straightedge and compass.

9. Find the minimal polynomial over $\mathbb{Q}$ for $\alpha = \sqrt{5} + \sqrt{7}$. Then explicitly describe the Galois group of the splitting field.

10. What is the degree of the extension $\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}$? Justify your answer.