Algebra Masters Exam, January 2014

Answer all questions. Partial credit will be given.

1. Let $G$ be a finite group with center $Z(G)$. Prove that if $G/Z(G)$ is cyclic, then $G$ is abelian.

2. Prove that the center of a finite $p$-group is nontrivial.

3. Consider the symmetric group $S_n$.

   (a) How many conjugates are there of $(1,2)$ in $S_n$ (for $n \geq 2$)?
   (b) How many conjugates are there of $(1,2,3)$ in $S_n$ (for $n \geq 3$)?
   (c) How many conjugates are there of $(1,2)(3,4)$ in $S_n$ (for $n \geq 4$)?

4. Let $S$ be the ring $M_2(\mathbb{R})$ of $2 \times 2$ matrices over the real numbers $\mathbb{R}$. Show that $S$ has exactly 2 two-sided ideals; namely, $(0)$ and $S$.

5. Find a subring of $M_2(\mathbb{R})$ isomorphic to $\mathbb{C}$, and give an explicit isomorphism.

6. Show that a finite integral domain (with $1 \neq 0$) is a field.

7. Prove the Steinitz exchange lemma: If $\{v_1, \ldots, v_m\}$ are linearly independent vectors in a vector space $V$, and $\{w_1, \ldots, w_n\}$ span $V$, then $m \leq n$ and (after reordering the $w_i$ if necessary) the set $\{v_1, \ldots, v_m, w_{m+1}, \ldots, w_n\}$ spans $V$.

8. Show how to construct $\sqrt{2} + \sqrt{2}$ with straightedge and compass.

9. Find the minimal polynomial over $\mathbb{Q}$ for $\alpha = \sqrt{5} + \sqrt{7}$. Then explicitly describe the Galois group of the splitting field.

10. What is the degree of the extension $\mathbb{Q}(\cos(2\pi/7))/\mathbb{Q}$? Justify your answer.