Starting academic year 2014/5 we will be offering 372 in fall semester and 473 in winter semester (but not 372). This necessitates a slight change in how the MS Algebra exam will be administered. In fall 2015 the MS Algebra exam will have 11 questions. Of these the student will answer nine questions from the Topics 1, 2, 3 and 4(a)(b)(c) listed below. They will then have a choice of answering a question on Galois theory (5(a)(b) below) or a question from Representation theory of finite groups, where the topics for this area are given in 6 (a)–(e).

Sample questions are given below.

1. Linear Algebra (3 questions)
   (a) Matrix algebra, determinants
   (b) Vector spaces
   (c) Linear transformations (change of basis, rank-nullity theorem)
   (d) Inner product spaces (Gram-Schmidt orthogonalization)
   (e) Eigenvalues and eigenvectors (characteristic polynomials, diagonalization, spectral theorem for symmetric or Hermitian matrices)

2. Groups (3 questions)
   (a) Important examples of groups (permutation groups, cyclic groups, dihedral groups, matrix groups)
   (b) Subgroups, normal subgroups
   (c) Homomorphisms (cosets, quotient groups, automorphisms), Isomorphism Theorems
   (d) Group actions
   (e) Class equation, Sylow Theorem(s) and applications

3. Rings (2 questions)
   (a) Ideals
   (b) Units
   (c) Homomorphisms, Isomorphism Theorems
   (d) Quotient rings
   (e) Prime and maximal ideals
   (f) Euclidean domains
   (g) Principal ideal domains; unique factorization
   (h) Polynomial rings
   (i) Chinese remainder theorem
4. Fields (1 question)
   (a) Fields of fractions
   (b) Finite degree field extensions and roots of polynomials
   (c) Finite fields

5. Galois Theory (1 question)
   (a) Cyclotomic extensions and cyclotomic polynomials
   (b) Fundamental theorem of Galois theory and applications

6. Representation Theory (1 question)
   (a) FG modules
   (b) Maschke’s theorem, Schur’s Lemma
   (c) Characters and character tables
   (d) Group Algebra
   (e) Inner products.

References

Sample Questions

Linear Algebra

1. If $A$ is an orthogonal square matrix, prove that all eigenvalues of $A$ have absolute value 1.

2. Are the matrices $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ similar? Explain.

3. Let $h : V \to W$ be a linear transformation, where $V, W$ are vector spaces with $V$ of finite dimension. Prove that $\text{dim}(V) = \text{dim}(\text{Image}(h)) + \text{dim}(\text{kernel}(h))$.

4. The matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ has eigenvalues 1, 2. Determine whether or not $A$ is diagonalizable.

5. Let $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, let $A$ be an $m \times n$ matrix, and let $y_i = Ax_i$, $i = 1, 2, \ldots, k$. Prove that if $y_1, y_2, \ldots, y_k$ are linearly independent in $\mathbb{R}^m$, then $x_1, x_2, \ldots, x_k$ are linearly independent in $\mathbb{R}^n$.

6. Let $P_3$ denote the collection of all polynomials of degree three or less with real coefficients, and let $T$ be the linear transformation $T : P_3 \to P_3$ defined by $T(f(x)) = f(x) + f'(x) + f''(x)$ where $f(x) \in P_3$ and where the prime denotes differentiation. Determine the matrix for $T$ relative to the basis $\{1 + x, 1 - x, x^2, x^3\}$.

7. Show that if $A$ is a diagonalizable real matrix with non-negative eigenvalues, then there is a matrix $S$ such that $S^2 = A$.

Groups

1. Determine the last 3 digits of the number $17^{2006}$. Explain your method.

2. Prove that there is no simple group of order 99.

3. Prove that any finite integral domain is a field.

4. Let $G$ be a group and let $a \in G$ have order $m$. Suppose that $a^s$ is the identity of $G$. Prove that $m | s$.

5. Prove that a group of order 42 must have a normal subgroup of order 21.

6. Let $\phi : G \to H$ be a group homomorphism. Assume that $G$ is an infinite group, that the kernel of $\phi$ is finite, and that the image of $G$ contains an element of order $p$ where $p$ is prime. Show that $G$ contains an element of order $p$. 


Rings

1. If $P$ is a prime ideal in a commutative ring $R$, prove that the set $P \times P$ is an ideal in $R \times R$. Is it a prime ideal? If so, prove it; if not, prove it or give a counterexample.

2. Let $R$ be the subring of $\mathbb{Q}$ consisting of fractions, which, when written in lowest terms, have denominator not divisible by $p$, where $p$ is a fixed prime. (You need not verify that $R$ is a subring of $\mathbb{Q}$.)
   
   (a) Show that $R$ has a unique maximal ideal $M$.
   
   (b) Determine $R/M$.

3. Let $f : A \to B$ be a homomorphism of rings with unit elements $1_A$ and $1_B$. Show that if $A$ is a field, then $f$ is either trivial ($f(a) = 0$ for all $a$ in $A$) or $f$ is injective (one-to-one).

4. (a) Prove carefully that the rings $\mathbb{Q}[x]/(x^3 - 2)$ and $\mathbb{Q}[x]/(x^2 - 3)$ are not isomorphic.
   
   (b) Find an example of a commutative ring with unity $R$ such that $R[x]/(x^2 - 2)$ and $R[x]/(x^2 - 3)$ are isomorphic. Justify your answer briefly.

5. Prove that the ring $\mathbb{Z}[x]$ (polynomials with integer coefficients) is not a principal ideal domain.

Fields and Galois Theory

1. Let $F$ be the splitting field of $x^3 - 3$ over the rationals. Find a basis for $F$ as a vector space over $\mathbb{Q}$, and prove your answer is correct.

2. Let $F$ be a field with 81 elements. Does the polynomial $x^3 - x + 1$ have a root in this field? (The polynomial should be considered as having coefficients in $\mathbb{Z}_3$.)

3. Construct a field with 8 elements and compute its Galois group over $\mathbb{Z}_2$.

4. Let $K$ be a splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2$. Justify carefully the answers to the questions below.
   
   (a) Find the degree of $K/\mathbb{Q}$.
   
   (b) Find the Galois group of $K/\mathbb{Q}$.
   
   (c) Find all intermediate fields between $K/\mathbb{Q}$ and indicate which ones are Galois over $\mathbb{Q}$.

Representation Theory

1. Let $V$ be a finite-dimensional $\mathbb{C}G$-module, where $G$ is a finite group. If $U$ is a submodule of $V$, show that there is a submodule $W$ of $V$ such that $V = U \oplus W$.

2. Determine the character table for the dihedral group of order 8.
3. Show that the kernel and image of a \( \mathbb{C}G \)-module homomorphism is a \( \mathbb{C}G \)-submodule.

4. Show that every irreducible representation of an abelian group has dimension 1.

5. Find the dimensions of all the irreducible representations of the dihedral group of order 10.

6. If \( \chi \) is a complex character of a finite group \( G \) and \( g \in G \) show that \( \chi(g) \) is a sum of roots of unity, and that \( \chi(g^{-1}) = \overline{\chi(g)} \).

7. Define the usual inner product \( \langle \cdot, \cdot \rangle \) on the set of characters of a finite group \( G \). If \( \chi \) is a character, show that \( \chi \) is an irreducible character if and only if \( \langle \chi, \chi \rangle = 1 \).