Algebra Exam Topics

Updated August 2017

Starting Fall 2017, the Masters Algebra Exam will have 14 questions. Of these students will answer the first 8 questions from Topics 1, 2, and 3. They then have a choice of answering two more questions from Topics 4, 5, and 6. Topic 6 has been added to accommodate ACME students who typically take Math 371 but not Math 372 nor Math 473.

1. Linear Algebra (3 questions)
   (a) Matrix algebra, determinants
   (b) Vector spaces
   (c) Linear transformations (change of basis, rank-nullity theorem)
   (d) Inner product spaces (Gram-Schmidt orthogonalization)
   (e) Eigenvalues and eigenvectors (characteristic polynomials, diagonalization, spectral theory for symmetric or Hermitian matrices)
2. Groups (3 questions)
   (a) Important examples of groups (permutation groups, cyclic groups, dihedral groups, matrix groups)
   (b) Subgroups, normal subgroups
   (c) Homomorphisms (cosets, quotient groups, automorphisms), Isomorphism Theorems
   (d) Groups actions
   (e) Class equation, Sylow Theorems and applications
3. Rings (2 questions)
   (a) Ideals
   (b) Units
   (c) Homomorphisms, Isomorphism Theorems
   (d) Quotient rings
   (e) Prime and maximal ideals
   (f) Euclidean domains
   (g) Principal ideal domains; unique factorization
   (h) Polynomial rings
   (i) Chinese remainder theorem

4. Fields and Galois Theory (2 questions)
    (a) Field of fractions
    (b) Finite degree field extensions and roots of polynomials
    (c) Finite fields
    (d) Cyclotomic extensions and cyclotomic polynomials
    (e) Fundamental Theorem of Galois Theory and applications
5. Representation Theory (2 questions)
    (a) FG modules
    (b) Maschke's Theorem, Schur's Lemma
    (c) Characters and character tables
    (d) Group Algebra
    (e) Inner Products
6. Advanced Spectral Theory (2 questions)
    (a) Generalized Eigenvectors and the Resolvent
    (b) Spectral Decomposition and Spectral Mapping Theorem
    (c) Perron-Frobenius Theorem
    (d) Drazin Inverse
    (e) Krylov Subspaces and GMRES

Sample Questions

Linear Algebra

1. If A is an orthogonal square matrix, then prove that each eigenvalue of A has absolute value 1.

2. Are the matrices $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ similar? Explain.

3. Let $h : V \to W$ be a linear transformation, where V and W are vector spaces with V of finite dimension. Prove that $\dim(V) = \dim(\text{Image}(h)) + \dim(\text{kernel}(h))$.

4. The matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ has eigenvalues 1 and 2. Determine whether or not A is diagonalizable.

5. Let $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, let A be and $m \times n$ matrix, and let $y_i = Ax_i$ for $i = 1, 2, \ldots, k$. Prove that if the vectors $y_1, y_2, \ldots, y_k$ are linearly independent in $\mathbb{R}^m$, then the vectors $x_1, x_2, \ldots, x_k$ are linearly independent in $\mathbb{R}^n$.

6. Let P_3 denote the collection of polynomials of degree 3 or less with real coefficients, and let T be the linear transformation $T : P_3 \to P_3$ defined by

$$T(f(x)) = f(x) + f'(x) + f''(x)$$

   where $f(x) \in P_3$ and where the prime denotes differentiation. Determine the matrix for T relative to the basis $\{1 + x, 1 - x, x^2, x^3\}$ of P_3.

7. Show that if A is a diagonalizable real matrix with non-negative eigenvalues, then there is a matrix S such that $S^2 = A$.

Groups

1. Determine that last three digits of 17^{2006}. Explain your method.

2. Prove that there is no simple group of order 99.

3. Prove that any finite integral domain is a field.

4. Let G be a group and let $a \in G$ have order m. Suppose that a^s is the identity element of G. Prove that $m \mid s$.

5. Prove that a group of order 42 must have a normal subgroup of order 21.

6. Let $\phi : G \to H$ be a group homomorphism. Assume that G is an infinite group, that the kernel of ϕ is finite, and that the image of G contains an element of order p where p is prime. Show that G contains an element of order p.

Rings

1. If P is a prime ideal in a commutative ring R, prove that $P \times P$ is an ideal in $R \times R$. Is $P \times P$ a prime ideal? If so, prove it. If not, give a counterexample.

2. Let R be a subring of $\mathbb{Q}$ consisting of fractions, which, when written in lowest terms, have denominators not divisible by p, where p is fixed prime. (You need not verify that R is indeed a subring of $\mathbb{Q}$.)

   (a) Show that R has a unique maximal ideal M.

   (b) Determine R/M.

3. Let $f : A \to B$ be a homomorphism of rings with units 1_A and 1_B. Show that if A is a field, then f is either trivial ($f(a) = 0$ for all $a \in A$) or f is injective (one-to-one).

4. (a) Prove carefully that the rings $\mathbb{Q}[x]/(x^3 - 2)$ and $\mathbb{Q}[x]/(x^2 - 3)$ are not isomorphic.

   (b) Find an example of a commutative ring with unity R such that $R[x]/(x^3 - 2)$ and $R[x]/(x^2 - 3)$ are isomorphic. Justify your answer briefly.

5. Prove that the ring $\mathbb{Z}[x]$ (polynomials with integer coefficients) is not a principal ideal domain.

Fields and Galois Theory

1. Let F be the splitting field of $x^3 - 3$ over the rationals. Find a basis for F as a vector space over $\mathbb{Q}$, and prove your answer is correct.

2. Let F be a field with 81 elements. Does the polynomial $x^3 - x + 1$ have a root in F? (The polynomial should be considered as having coefficients in $\mathbb{Z}_3$.)

3. Construct a field with 8 elements and compute its Galois group over $\mathbb{Z}_2$.

4. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2$. Justify carefully the answers to the questions below.

   (a) Find the degree of $K/\mathbb{Q}$.

   (b) Find the Galois group of $K/\mathbb{Q}$.

   (c) Find all intermediate fields between K and $\mathbb{Q}$, and determined which of these fields are Galois over $\mathbb{Q}$.

Representation Theory

1. Let V be a finite dimensional $\mathbb{C}G$-module, where G is a finite group. If U is a submodule of V, show that there is a submodule W of V such that $V = U \oplus W$.

2. Determine the character table for the dihedral group of order 8.

3. Show that the kernel and image of a $\mathbb{C}G$-module homomorphism is a $\mathbb{C}G$-module.

4. Show that every irreducible representation of an abelian group has dimension 1.

5. Find the dimension of all the irreducible representations of the dihedral group of order 10.

6. If χ is a complex character of a finite group G and $g \in G$, show that $\chi(g)$ is a sum of roots of unity and that $\chi(g^{-1}) = \overline{\chi(g)}$.

7. Define the usual inner product $\langle \cdot, \cdot \rangle$ on the set of characters of a finite group. If χ is a character, show that χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Advanced Spectral Theory

1. Prove that the resolvent $R(z)$ satisfies the equation

$$\frac{d}{dz}R(z) = -[R(z)]^2.$$

2. Let A be a square matrix. Let $\lambda \in \sigma(A)$ and let P be the associated spectral projection. Prove that if $0 \notin \sigma(A)$, then A^{-1} exists, λ^{-1} is an eigenvalue of A^{-1} and P is the spectral projection of A^{-1} associated with λ^{-1}.

3. Obtain the spectral decomposition of the matrix

$$A = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

4. Suppose $V = X \oplus Y$ and let P be the projector onto X along Y. Prove that

$$\mathscr{R}(P) = \mathscr{N}(I - P) = X \quad \text{and} \quad \mathscr{R}(I - P) = \mathscr{N}(P) = Y.$$

5. Prove that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are vectors such that $\mathbf{v}^T\mathbf{u} = 1$, then

$$||I - \mathbf{u}\mathbf{v}^T||_2 = ||\mathbf{u}\mathbf{v}^T||_2 = ||\mathbf{u}||_2||\mathbf{v}||_2 = ||\mathbf{u}\mathbf{v}^T||_F.$$

6. Compute the eigenvalues, resolvent, and spectral projections of the matrix

$$\begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

7. Find the Laurent expansion of the matrix in the problem 6 around $z = 2$.

8. Let B be the matrix in problem 6

   (a) Write the spectral decomposition $B = \sum_\lambda \lambda P_\lambda + D_\lambda$.

   (b) Write the resolvent in the form of (11.31).

   (c) Give four different square roots of B.

   (d) Compute $\sin(B\pi/4)$.

9. Compute the Drazin inverse of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$

10. Compute the spectral radius of

$$\begin{bmatrix} 9 & -1 & 2 \\ -2 & 8 & 4 \\ 1 & 1 & 8 \end{bmatrix}$$

11. Determine the Krylov subspace $\mathscr{K}_3(B, \mathbf{b})$ where

$$\mathbf{b} = (1\ 1\ 1\ 1)^T$$

12. Suppose that A is an $n \times n$ matrix with $A > 0$ and $\rho(A) = r$.

    (a) Explain why $\lim_{k\to\infty}(A/r)^k$ exists.

    (b) Explain why $\lim_{k\to\infty}(A/r)^k = G > 0$ is the projector onto $\mathscr{N}(A - rI)$ along $\mathscr{R}(A - rI)$.

    (c) Explain why $rank(G) = 1$.

13. Let $A \in M_n(\mathbb{F})$. Prove that if every row (or column) sum of $A > 0$ is equal to ρ then $\rho(A) = \rho$.