Master's Analysis Exam – February, 2015

3 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Let $\{a_n\}$ be a monotonically increasing sequence of real numbers. Prove that if $\{a_n\}$ is bounded above, then the sequence converges.

2. Prove that if a series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

3. Prove that there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for some point (a, b), f is not continuous at (a, b), but $\lim_{x\to a} f(x, a) = f(a, b)$ and $\lim_{y\to b} f(a, y) = f(a, b)$.

4. Prove that the set of invertible real $n \times n$ matrices is an open set in the metric space of all $n \times n$ matrices, where $d(A, B) = ||A - B|| = \sqrt{\sum_{i,j}(a_{i,j} - b_{i,j})^2}$.

5. Prove that $\nabla \times (\nabla f) = 0$ wherever f has continuous second partial derivatives.

6. Use the Implicit Function Theorem to analyze the solutions to the following system near the solution $(0, 0, 0)$.

$$\begin{cases} (x^2 + y^2 + z^2)^3 - x + z = 0 \\ \cos(x^2 + y^4) + e^z - 2 = 0 \end{cases}$$

7. Prove or disprove: let C by any simple smooth closed curve inside the unit circle of $\mathbb{R}^2$. Then

$$\oint_C x \tan y dx - x \sec^2 y dy = \oint_C (x + 1) \tan y dx.$$

8. Let $f_n(x) \to f(x)$ uniformly in I, where $I \subset \mathbb{R}$ is some compact interval. Prove that if the f_n are integrable, then f is integrable, and

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f(x)dx.$$

[Hint: may be helpful to first prove that the functions f_n are uniformly bounded.]

9. Prove that a function, which is analytic in the whole complex plane and satisfies the inequality $|f(z)| < |z|^n$ for some n and for all sufficiently large $|z|$, is a polynomial.

10. Compute $\displaystyle\int_0^{\infty} \cos(x^2)dx$.